Notes on Projections

Part I - Categories and Properties

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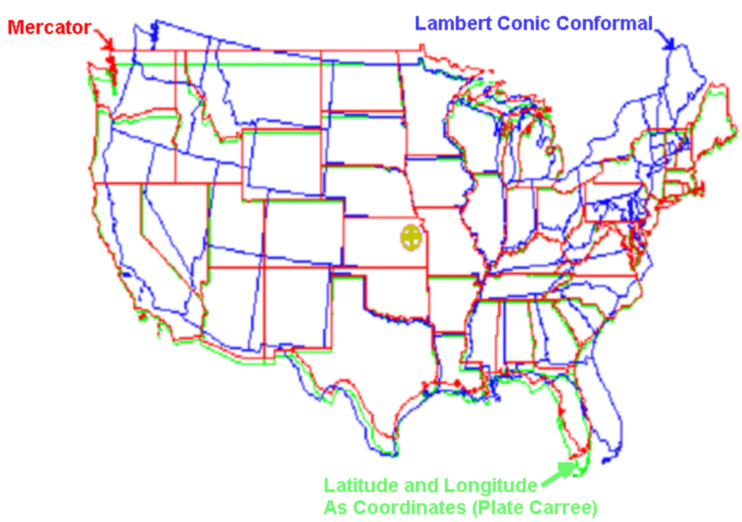
I. Introduction and References

The world is, approximately, a sphere. Maps are flat. Making maps requires some method of putting the geographic information on the flat piece of paper. This is called the **projection**. There are approximations and distortions introduced in this process. The many different projections are different choices trading off the distortions. There is no one best projection, different projections are better in some situations than others.

Projections have different properties or characteristics. They may also be lumped into families and categories. Each particular projection usually has a small number of parameters that must be specified. This can be the center of the map, the direction of the vertical on the map etc. Given an area such as the United States, there are many projections that do a "fairly good" job. Of course what is good usually depends on the application.

A map of the US is shown below in three different projections. The maps are all the same at the central US point marked with a circle. They all have the same nominal scale. The red map is a Mercator projection. The blue is a Lambert Conic Conformal. Both these are common. The third uses latitude and longitude directly to plot the lines (this is officially called a Plate Carree). This is common in simple computer plots.



US in Three Map Projections
All Centered at 39 N 96 W

Both the Mercator and Lambert are angle preserving, or in the nomenclature of projections conformal. But in the large, they do not have quite the same shape. One is based on a cylinder the other on a cone as a projection surface. But they don't look that different. So "vastly different" projections can both be fairly good at representing the earth. It is just a matter of choosing the parameters that define the projection.

Understanding the characteristics of the projections, and to some extent the choice of parameters is the purpose of this note. For actual map making the details are complex. For this case some reference works are needed.

There are three reference works that the a user should know about on the topic of projections. First, the standard for the navigator, Bowditch should be consulted. This is officially *The American Practical Navigator*, 1995, Publication Number 9, NIMA. It is printed by the US Government Printing Office. Electronic copies are now available. Projections are covered in Section I, Chapters 1 to 4. These chapters give a quick, concise overview. Many important terms are in bold print. Most bold items are listed in the extensive glossary.

A detailed user's reference manual on projections and datums is given in the NIMA Technical Manual 8358, *Datums, Ellipsoids, Grids and Grid Reference Systems*, 1990, NIMA. This is also available electronically. Many examples and explanations of specific map uses and notations are given in this publication.

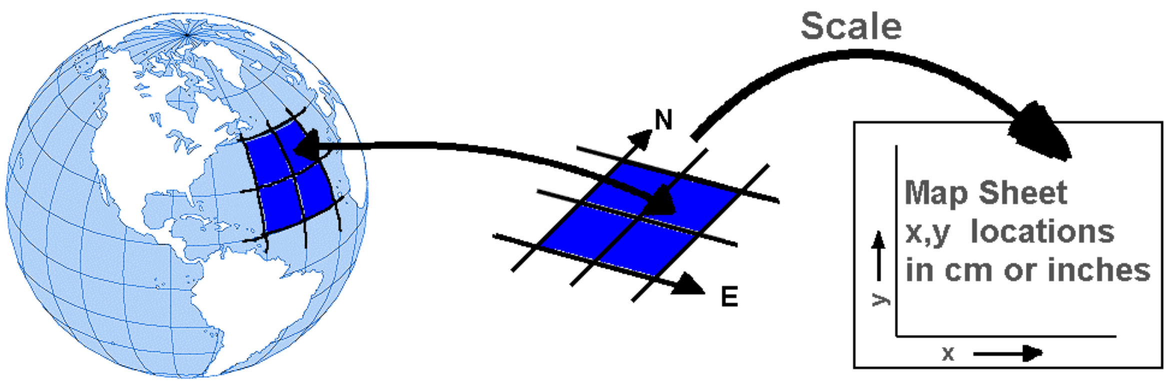
Finally, the definitive source for the mathematics of a very large number of projections is given in the US Geological Survey Professional Paper 1365, *Map Projections, A Working Manual*, by John P. Snyder, 1987, US Government Printing Office. (Others also sell Snyder.) Snyder gives an overview of map projection characteristics, in full detail. He follows this with the details of 25 families of projections. Each projections has a small history, a set of equations for the sphere, a set for the ellipsoid and notes on characteristics and uses. If you must program projections into a computer, this is the ideal reference. Excepts from Snyder's introduction are available on line.

Note well, if you want to take coordinates off a map, the projection is important, but not the whole story. You have to also know the ellipsoid and datum used. You need to look at the map legend.

II. Categories of Projections

There are several ways to categorize map projections. The different methods overlap and some distinctions are not used consistently in texts.

At the most general level projections can be classified as being **usual** or **exotic**. The exotic category has come about with computers. These are usually projections used the entire world. They try to minimize distortion through different techniques such as the "flattened orange peel" look of some. A few exotic projections predate the computer age, but many more are now available. Projections that are on common maps will be discussed here.



Mapping Round Earth
On Flat Paper Leads to Distortion

A second way to categorize map projections is as **geometric** or **mathematical projections**. In geometric projections, maps can be constructed with rulers, compasses, etc. A physical map can be made without a computer. Geometric projections are further divided into **Perspective** and **Non-Perspective projections**. The mathematical projections often have no construction basis. Clearly the mathematical equations for the geometric equations exist, and can be found in books like Snyder. Some of these were developed at the time the projection was introduces. For the older non-perspective geometric projections, such as the Mercator, the equations only became available decades after they were introduced.

In the usual map the north direction up and eastward to the right. But this is not always the case. If we define the paper locations with a usual x-y Cartesian coordinate system, then the mapping or projection is defined by two functions:

$$x = f(\phi, \lambda)$$
$$y = g(\phi, \lambda) .$$

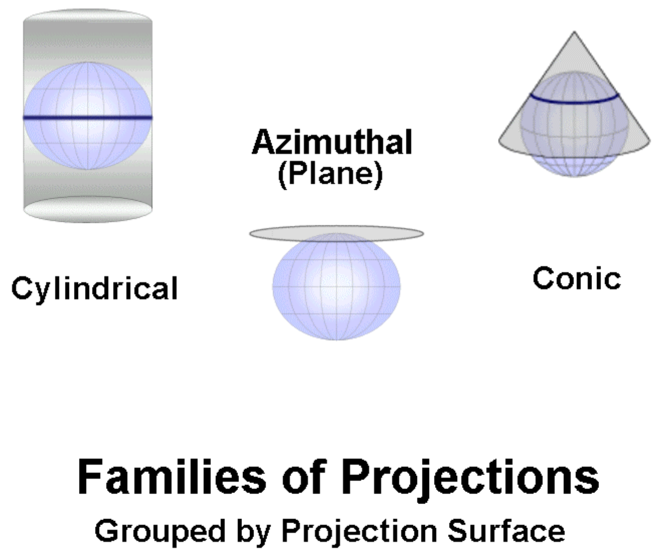
This what a mathematician would call a mapping.

Within the generic geometric classification are **prospective projections**. In these projections, the map looks as like the earth seen from some particular place. This place can be the center of the earth, the opposite side of the earth (assuming the earth is transparent) or far out in space. In another description of prospective projections, some surface is positioned near, on or in the earth and rays go through the earth leaving a shadow on the surface. In this description the origin of the rays replaces the observer.

A final view on projections describes how adjacent maps are generated and fit together. If a new projection is used for each map the edges of maps will not line up. The same general projection may be used for the two adjacent maps, but choosing different parameters make for different projections. Some map projections have a fixed projection over a very large area (although not the entire world). Then individual maps are just "windows" into this larger map. The edges line up in this case. The most common military projection, the Universal Transverse Mercator is a **windowed projection**.

III. Map Families

The surface placed "near, on or in" the earth described above is usually a plane, a cone, or a cylinder. These are called the three families of map projections. Even if the projection is purely mathematical, it is often placed in one of these three families, **cylindrical, conic and azimuthal (planar).**



The various types of projections are made on these surfaces while they are in place. Then the surface is cut (except for the plane) and rolled flat. For a cone the cut is from the tip to the base along a straight line. For the cylinder the cut is along the "side" of the can. These three surfaces, plane, cone, and cylinder are called **developable surfaces**.

Notice that the plane is also called the azimuthal projection. This is because the azimuth from the center points is true. In map projections terminology, azimuthal and planar are the same.

IV. Projection Characteristics

A. Desired Attributes of Maps

There are several important attributes that we would like the projection to have. Unfortunately we cannot define a projection that has them all at the same time. The most important characteristics we would like are:

Shape The shape of things is preserved,

Direction	Angles measured on the map are the same as those on the ground,
Scale	Distances measured on the map times the scale are correct, and
Area	Areas measured on the map are correct.

The key to all of these is the behavior of the scale function. On most projections the ratio between the true scale and the nominal scale varies with position, and it is also a function of the direction you look at each point. The formulae ratio the actual to true scale in the two principle directions, north-south and east-west, are commonly given in manuals on projections. In practice these two values, called h and k, are sufficient to define all the above characteristics. (Mathematically it takes three parameters to define the scale ratios for a general projection. The third, which represents the orientation of the distortion is not usually given.)

In the study of projections you can encounter several levels. You will also encounter geometric descriptions of projections in elementary books. At a more detailed level you need to know about the scale parameters and how they determine the properties of shape, area, direction and scale. And finally for specific projections you can have two levels of mathematics on the actual mapping, the f and g functions. The simple form are the formulae for a sphere. The more complex are the formulas for an ellipsoid.

If you wish to use a map for detailed navigation or location finding, you should have a basic knowledge of the characteristics of the projection you are dealing with. If you wish to make a map, to produce a computer program to do a projection, there are several books that have a catalogue of projections and the formulas for a sphere and for an ellipsoid. One of the best is *Map Projections - A Working Manual*, by John Snyder.

B. The Two Major Properties

A **Conformal** projection preserves angles. Therefore directions are true, and in the small shapes are correct. Most projections used for navigation are conformal. The Mercator projection is conformal. In fact it was designed to be conformal to make it useful for navigation. (Mercator became a big seller of maps from his invention of this projection.) The other most common navigation projection, the Lambert Conic Conformal projection includes this property in its name.

A map that preserves area is called **Equal Area**. The Adler's Equal Area projection has this property. This does not mean that shapes are correct, just that areas are. The distortion in different directions cancel when you compute area. The stretching in one direction is cancelled by shrinking in the perpendicular direction at each point. The product of measurements in two perpendicular directions is correct.

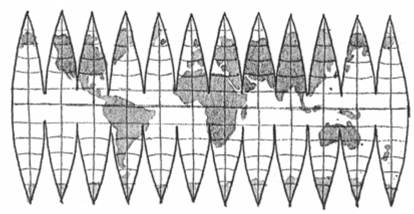
Unfortunately you cannot have a map be both Conformal and Equal Area. The processing of flattening the world requires choices and compromises.

C. Other Important Terms and Properties

An **Equidistant** projection has the "latitude" lines evenly spaced. These are sometimes straight lines and sometimes curved. (The word latitude is in quotes because for an oblique projection what would normally be latitude is some other curve on the earth.) Plotting the map directly in the variables of latitude and longitude in degrees is common in simple computer displays. This is an **Equidistant Cylindrical projection**. This is also called a **Plate Carree**. (There are other equidistant cylindrical projections that have different scale factors on the two axis.) If you make a simple polar plot of values in latitude and longitude, you generate an **Equidistant Azimuthal projection**. Here the lines of latitude are evenly spaced circles.

A **standard line** is a line (not necessarily straight) along which the scale is true. Some projects have families of standard lines, some none. If a projection comes from one of the three developable surfaces, the lines or points where the surface touch the earth are usually standard lines. For standard equidistant plots (non-oblique), the meridians lines are standard lines. The north-south distortion parameter, h, is one.

An **interrupted projection** looks like it is a globe that has been but out in sections. The simplest version, is the "orange peel" version of a globe.



Orange Peel of Globe
(Interrupted Sinusoidal Projection)

This version is not used except to illustrate the problems of flattening a globe to make a map. However there area several very common interrupted projections, such as the Goode.

V. Scale Error and Map Properties

The scale of a map is the nominal ratio of the distance on the map to the distance on the ground. This is often called the **representative fraction**. It is written as 1:24,000 which is read as one to twenty four thousand. It is important to note the word "nominal" above. The scale will be correct at most along a line or a series of lines. The difference between the true scale and the nominal scale is the scale distortion.

On projections the nominal scale will only be correct at a point on along a few lines. The error at each point will also be a function of direction. It is customary to define the scale error in the principle directions north-south as **h** and east-west as **k**.

$$h \equiv \frac{\text{North} - \text{South Scale}}{\text{Nominal Scale}}$$
$$k \equiv \frac{\text{East} - \text{West Scale}}{\text{Nominal Scale}}.$$

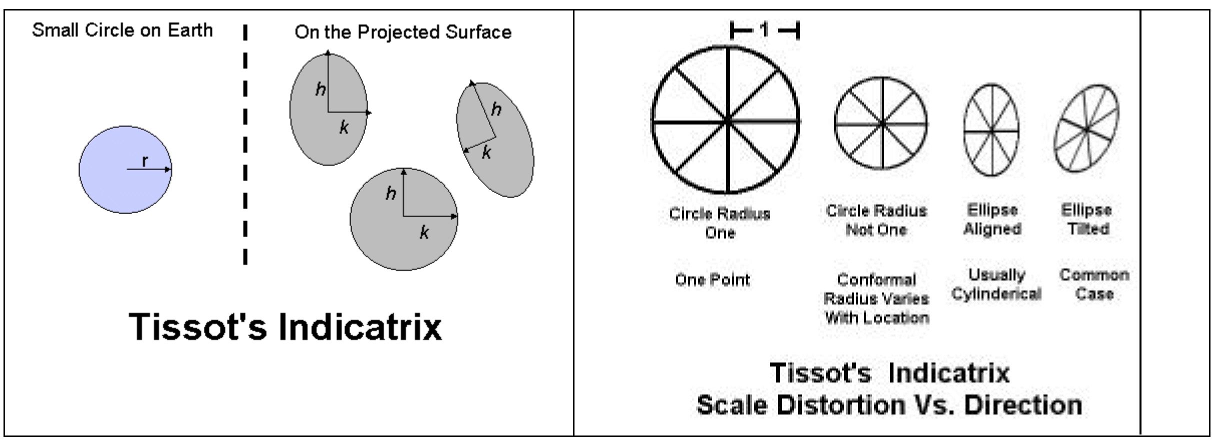
The values of h and k will be a function of position and direction[*]. If h is one, then the scale is true at that point in the north-south direction. In general h and k are functions of latitude, and sometimes also of longitude. These functions are given in advanced projection books such as Snyder.

The major characteristics of a projection can be defined in terms of h and k. For example if you want angles to be true, that is the map to be **conformal**, then you need to have **h = k everywhere**. That does not mean that h and k do not vary, just that they vary together.

For example a Mercator projection is formed from a cylinder wrapped around the earth. The projection "point" is the center of the earth. The meridians will be straight vertical lines. On the earth the meridians meet at the poles. Therefore there is increasing distortion in the east-west direction as you go away from the equator. That is k increases. Mercator chose a particular form for the latitude function (f above) that had the same distortion. Today we express this as a function. Earlier it was done geometrically where the projection point of each latitude varied. It was only the center of the earth for the equator.

If you wish to have the area to be true, an equal area projection, you need the product h*k = 1 everywhere. It is not possible to have both an equal area and conformal map.

I you plot the value of the distortion as a function of direction at some point, you get an ellipse. The distortion is taken as the distance from the center. This type of diagram is call **Tissot's Indicatrix**. There will be a different ellipse at each point on the map.



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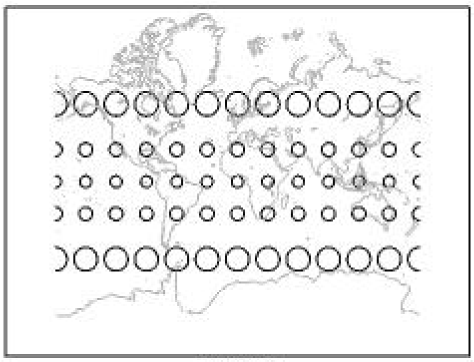
The formal mathematical definition of h and k involves the partial derivatives of the map coordinates, x and y, with respect to the latitude, ϕ, and longitude, λ in radians (natural angle units). For a globe of radius R (or R = Scale * Earth-Radius) then,

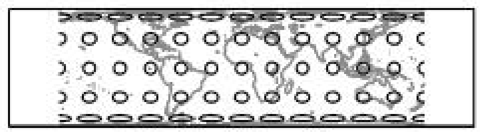
$$h = \frac{1}{R}\frac{dy}{d\phi}$$

$$k = \frac{1}{R\cos\phi}\frac{dx}{d\lambda}.$$

Two examples of Tissot's Indicatrix plotted on world maps are shown below. The first is for the Mercator projection. It is the most common projection used for world maps. The indicatrix diagrams are all circles, but the sizes vary with latitude. The Mercator is a conformal projection. It is not an equal are plot. The second plot is for an equal area cylindrical projection. The longitude scale is the same for both. For the equal area plot the indicatrix diagrams are not circles but ellipses. The stretching in the north-south reduction has been reduced so that h*k is one at each latitude. This is not a conformal plot.



Mercator



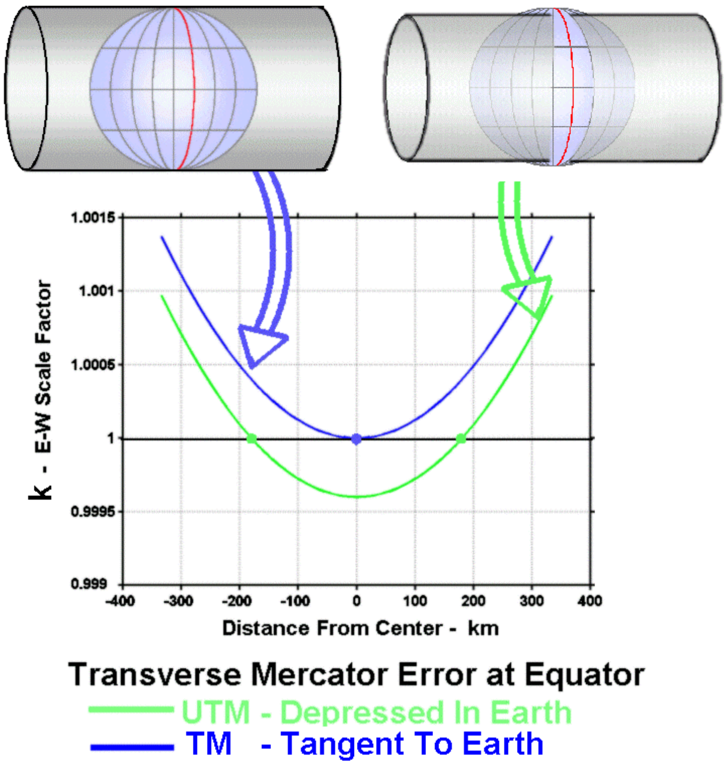
Cylindrical Equal Area

VI. Projection Error Control

For a developable surface tangent to the earth there is no distortion where the surface touches the earth. But the error grows away from that line (or point). It usually grows as the square of the distance from this location. One common strategy to control the error is to depress the surface into the earth. In the figure below a cylinder is placed on the earth with the tangent line being a meridian of longitude. This is the geometry for the transverse Mercator projection. On the right the cylinder is slightly smaller that the earth diameter and cuts the earth along two lines. This is the geometry for the Universal Transverse Mercator, or UTM, projection.

The scale factor for the tangent case is one only at the tangent point. For the depressed case the value of h is one along the two lines that cut the earth. Now the error, that is the difference of h from 1, is both positive and negative. The maximum deviation is less.

These are errors in making a map measurement, over a small distance, are the values of k-1. Large distance measurements need to add up the individual errors at each point. (That is integrate the curves below.) Because some of the values are negative and some positive the total error is much less than for the tangent case. While the error goes from - 3 km to 6 km, the error from one side of the map to the other along an east-west line is on 50 m. The same error without depressing the cylinder is 320 m,
6 times as large.



The line(s) where the surface cuts the earth have true scale and are **standard lines**. If these are toward the edge of the map (typically 10 to 20 percent from the edge) the maximum error will be much less. The two most common maps used for navigation, the Lambert Conic Conformal and the Universal Transverse Mercator use this technique. Military maps are UTM except in polar regions, where they are a version of the Stereographic that also uses this technique for error control.

VII References

1. NIMA Geodesy and Geophysics Online Reference Material
http://www.nima.mil/GandG/pubs.html

2. USGS Map Information Page
http://mac.usgs.gov/mac/isb/pubs/pubslists/index.html

3. Snyder, John P., *Map Projections, A Working Manual*, US Geological Survey Professional Paper 1365 , 1987, US Government Printing Office.

4. *Datums, Ellipsoids, Grids, and Grid Reference Systems*. Edition 1. US National Imagery and Mapping Agency (NIMA). TM 8358.1, Washington, D.C., NIMA, 20 September 1990. (Available at reference 1 website as a PDF file.)

5. *The Universal Grids: Universal Transverse Mercator (UTM) and Universal Polar Stereographic (UPS).* Edition 1. US National Imagery and Mapping Agency (NIMA). TM 8358.2 Washington, D.C., NIMA, 18 September 1989. (Available at reference 1 website as a PDF file.)