Datums - Map Coordinate Reference Frames

James R. Clynch
Naval Postgraduate School, 2002

I. What Are Datums – in Geodesy and Mapping?

The basic fact is: the **coordinates** of all locations on the earth are on **datums**. At high accuracy the coordinates of a point, the latitude, longitude and height or Cartesian coordinates, may exist in several self-consistent sets. These will be in different datums.

A datum is the answer to the practical problem of making an accurate map. If you wish to determine the relative location of a pair of points a few meters apart, the solution is obvious. Just measure the difference with a tape measure. The issue of orientation still exists though, but this can be solved using two "known" points to measure a third. Or observations of the stars can be used to define north.

In effect this defines a **local datum**. The known point, together with some method for determining the direction of north, defines the location of points measured from it. If the reference point is in error by 100 m north, so will all the points using it. They move together. This of course assumes these errors are small, at least as compared to the radius of the earth.

If you look at the legend of a topographic map, you will find that it lists the "datum" that is used. In fact there may be several datums, one for horizontal, one for vertical etc. These are important because they define the reference system that is used for the coordinates.

If you use a GPS navigation system not set to the map datum, you can be off by 100 m (usually) to a kilometer (sometimes). Navigation systems produce coordinates on datums. If this is not the same datum as the map being used, an error occurs. Several ship accidents have happened because the GPS navigation system and the navigation chart were on different datums.

The practical way to define a datum is with a whole set of reference markers and their associated coordinates. They should be carefully surveyed together. This gives a network that serves as a "**realization** of the datum". This provides a practical set of points spread out over the region covered. Surveyors use the closest survey marker that meets the accuracy needs. In practice almost all surveying is relative, from one point to the next.

> This means that datums are the reference frames used in the construction of maps. Realization, the disks in the ground and the catalogue of their coordinates, are the practical way datums are used.

Things are complicated in practice as the same area may have two datums giving each point two different coordinates. In addition datums were often generated by individual countries, and did not match at the boundaries. The 1900 era German maps of France and the English maps of Germany did not match, even in the areas of overlap.

In general if a datum covers areas not directly connected by a survey, such as over a body of water, there are really different versions of the datum. This is not an academic

point. At least one recent ship grounding in the Caribbean was caused by using the wrong "NAD 83" in a GPS receiver. They used a version of "NAD 83" that did not match the "NAD 83" on the chart.

II. Origin of Datums

When we see a map or a globe today we have a fairly good idea of the meaning of the latitude and longitude lines on it. The longitude lines have a zero in Greenwich England and the zero of latitude is the equator. Even today there are subtle difference in the meaning of these statement. In the past there were major differences depending on the country you were from. The coordinates were said to be on different datums.

In the past each major country had its own system of coordinates. The latitude was meant to be the same, basically defined by the spin axis of the earth. But the origin of longitude is completely arbitrary and each country chose a convenient location on its soil. Thus two maps that were of the same area would not have the same coordinates for a city.

In fact both the latitude and longitude would often be different, although the latitude difference would be small. These reference systems were set up by establishing a "good primary reference site" and then doing **relative surveying** outward from it. This was usually an astronomical observatory so the coordinates of this primary point would be well known from celestial observations. The outward surveying had to stop at oceans and other major obstacles. So islands frequently had their own primary reference point that was not as well determined as the ones in major European capital cities.

Because these systems were not quite consistent from country to country, even where there were no survey problems, maps differed based on which country made them. Each map set was on a different Datum.

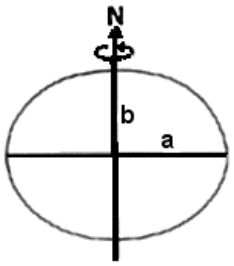
In reality there are many kinds of datums. Here we will only consider two, horizontal datums and vertical datums. Mapmakers distinguish between horizontal and vertical position because different techniques are used to measure heights and horizontal positions.

And things are even more complicated because the world is not quite a sphere.

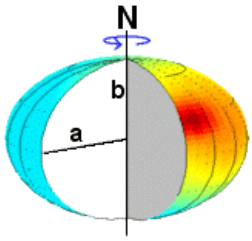
III. Datums and Ellipsoids

The world is slightly bigger at the equator than at the poles. The distance from the center of the earth to the equator is larger than the distance from the center to the poles by about 23 km. This is a factor of about 1/300 or 0.3 percent. While this is small, it is important for maps of the world or even much smaller regions.

The non-spherical earth is modeled by an ellipse of revolution. This shape is called and **ellipsoid**. (In some books it is called a **spheroid**.) In order to make an ellipsoid model of the earth, take an ellipse and align the shorter axis with the spin axis of the earth. The longer axis will point out the equator. Rotate the ellipse about the shorter, polar, axis to form a solid. This is the ellipsoid.



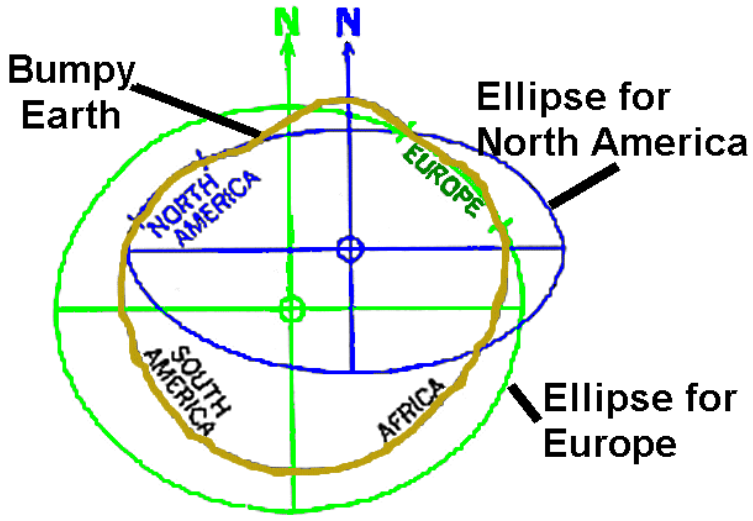
Rotate Ellipse



Making Ellipsoid

The shape and size of the ellipsoid representing the earth is very difficult to measure. A good measurement needs celestial observations at many points all over the world. These were unavailable before satellite based surveying began in the 1960's. So countries made due with what they had, usually over a "small region" like Europe.

Each country decided for itself what was the best estimate of the size and shape of the earth. They each had their own ellipsoid. And they also had their own primary reference point. Defining the latitude and longitude of a particular point on the earth defines the origin of the ellipsoid. That is choosing an ellipsoid and primary reference point coordinates gives both the shape and location of the ellipsoid. This model of the earth is needed for the surveying that defines the location of other points on maps.



Different Ellipsoids From Fitting Different Regions of Earth

Ellipsoids made from a "local" area's data, like Europe or North America could be significantly different. This did not bother the mapmakers because they just wanted something to use in making maps that worked well for their area. For example the ellipsoids chosen for Europe and North America are quite different, with the origins being offset about 250 m.

The ellipsoids are usually defined by two quantities. These could be the length of the two axes. However a more common set used in geodesy and surveying is the **semi-major axis** (equatorial axis length) and the **flattening**. The polar axis is also called the **semi-minor axis**. If the equatorial axis is called a, and the polar axis b, then the flattening is defined as

$$f = \frac{a-b}{a} = 1 - \frac{b}{a} .$$

A list of the most common ellipsoids in use is given in the following table.

Name	Semi-Major Axis -a (Km)	Semi-Minor Axis - b (km)	1/Flattening
Airy	6377.563	6356.257	299.32
Modified Airy	6377.340	6356.034	299.32
Australian National	6378.160	6356.775	298.25
Bessel 1841	6377.397	6356.079	299.15
Clarke 1866	6378.206	6356.584	294.98
Clarke 1880	6378.249	6356.516	293.46
Everest	6377.276	6356.075	300.80

Fischer 1960	6378.155	6356.773	298.30
Helmert 1906	6378.200	6356.818	298.30
Indonesian 1974	6378.160	6356.774	298.25
International	6378.388	6356.912	297.00
Krassovsky	6378.245	6356.863	298.30
South American 1969	6378.160	6356.774	298.25
WGS 72	6378.135	6356.751	298.26
GRS 80	6378.137	6356.752	298.257
WGS 84	6378.137	6356.752	298.257

A datum in the modern sense is defined by choosing an ellipsoid and then a **primary reference point**. Therefore giving the ellipsoid used is not enough. The **North American Datum of 1927, NAD27**, uses the Clarke 1866 ellipsoid and a point in central Kansas (called Meade's Ranch) as its primary reference point. Note that some maps of some Caribbean islands are also listed as being on NAD27. But this is really a different datum because a different primary reference point was used.

There was a major update of the North American Datum in the early 1980. This resulted in the **North American Datum of 1983 or NAD83**. NAD83 uses Meade's Ranch but a new ellipsoid. The ellipsoid is **GRS80**, which is the same as the WGS84 ellipsoid. (**The World Geodetic System of 1984, WGS84**, was established at about the same time by the US Defense Department.) In essence new data was used to establish better coordinates for the existing major benchmark network in North America.

Thus the brass disks in the ground are the same, only a new database of coordinates for each is different. The update was done with many new measurements and computations. But the main result was a new **catalogue of the coordinates** of the survey marks already in the ground.

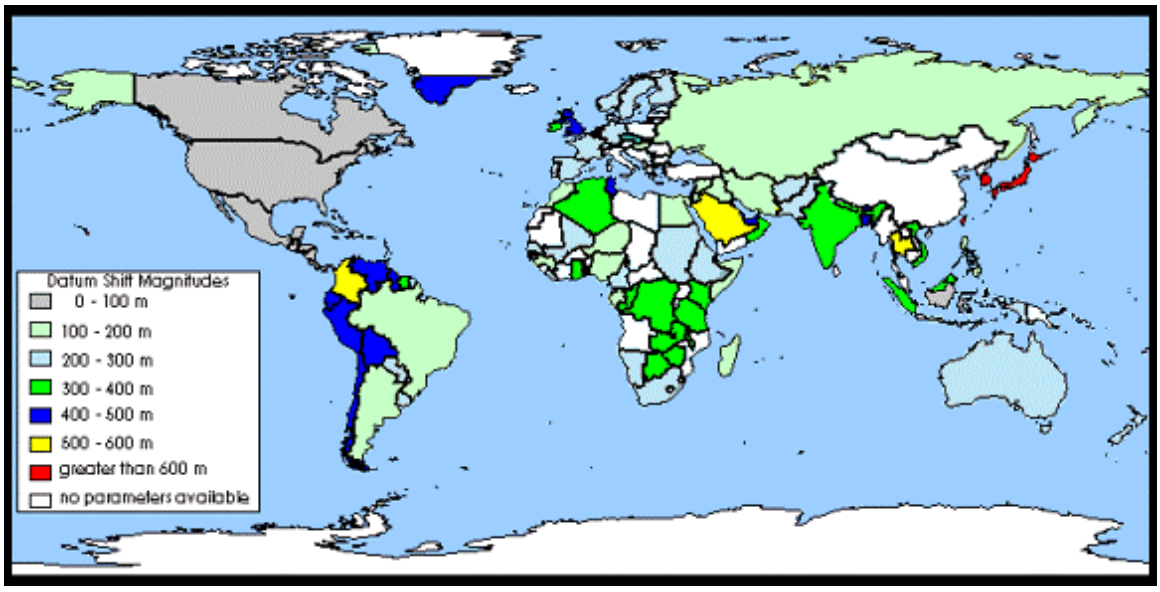
A few of the common datums in use today are:

North American Datum of 1927	NAD27	Clarke 1866
North American Datum of 1983	NAD83	Clarke 1866
North American Vertical Datum of 1929	NAVD29	GRS80 = WGS84
North American Vertical Datum of 1988	NAVD88	GRS80 = WGS84
World Geodetic System of 1984	WGS84	WGS84 = GRS80
South American Datum of 1969	SA1969	South American
European Datum of 1950	EU50	International

IV. Modern Datums and Reference Frames

Today maps almost always use Greenwich England for the longitude origin and try to have the best values for the ellipsoid and primary reference points. However we are still left with older surveys and maps that are on different datums. There are hundreds of them, but only about 50 that have common use.

With the advent of satellite surveying systems, worldwide coordinate systems were needed. This led to the establishment of worldwide systems. The US Defense Department made the first of these in 1966. A later one, called **World Geodetic System 1972, WGS72**, was quite successful. It was used for the Navy Navigation System, which was opened to public use. Later a better system, called WGS84 was generated when the **Global Positioning System (GPS)** required better coordinates.

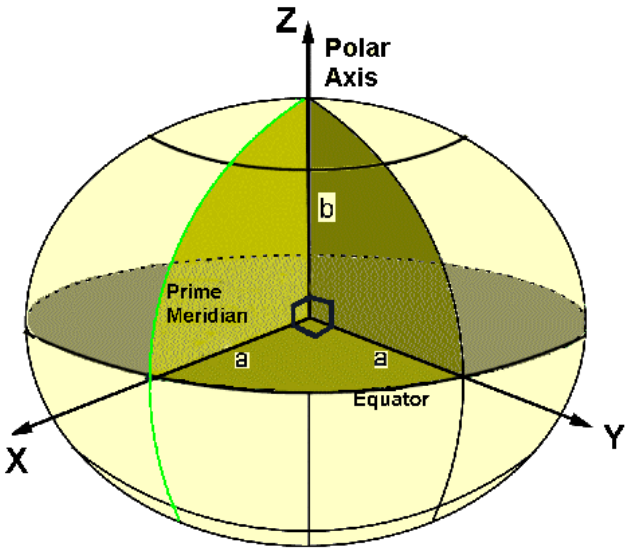


The above diagram shows the approximate shift in horizontal location between WGS84 and datums used commonly for maps. While the US is now officially on the North American Datum of 1983 (NAD83), which is essentially identical to WGS84, there are still many maps on the older NAD27. This is true even of government printed topographic maps from the US Geological Survey. In other areas of the world the shift are even larger. Kilometer size shifts exist for the Tokyo datum for example.

The science community has been working on a series of world reference systems that are called **International Terrestrial Reference Systems or ITRF's**. The earliest ones were ITRF92 and ITRF94, which was quite good. Modest improvements followed with ITRF97 and ITRF2000. The later two models were so accurate that models of the motion of the **crustal plates** of the earth had to be included.

The WGS's and ITRF's were basically defined as **Cartesian XYZ systems**. These are a perpendicular axis with the origin at the center of the earth. The Z-axis goes out the

north pole. The X-Y plane is the equatorial plane. The positive X-axis defines the origin of longitude. This is called an **Earth Centered, Earth Fixed, ECEF coordinate** system. An ellipsoid was associated with each. This was needed to convert the xyz coordinates to latitude, longitude and height. Of course height had it's own complication which are discussed elsewhere.



Earth Fixed Cartesian Coordinates

X-Y Plane is Equatorial Plane
X On Prime Meridian
Z Polar Axis

V. Realizations

A. How Datums Really are Used

There is a second whole side to datums, the practical side. The primary reference point for the US is in central Kansas at a place called Meade's Ranch. It is impractical to begin all surveys in Kansas. The mapping organization of each nation do high quality surveys and establish a network of high accuracy points. These are usually a bronze disk set in concrete or rock.

This disk, along with the coordinates of the small punch mark in the disk center, generates a "realization" of the datum.

B. Pre-Satellite Era Realizations

In the past this work involved measuring angles between points. This was done in a series of triangles or more often four sided figures with the internal diagonals measured. These are called **braced quadradrils**. The primary network of the US prior to satellite surveying begins with a series of these in a large network.

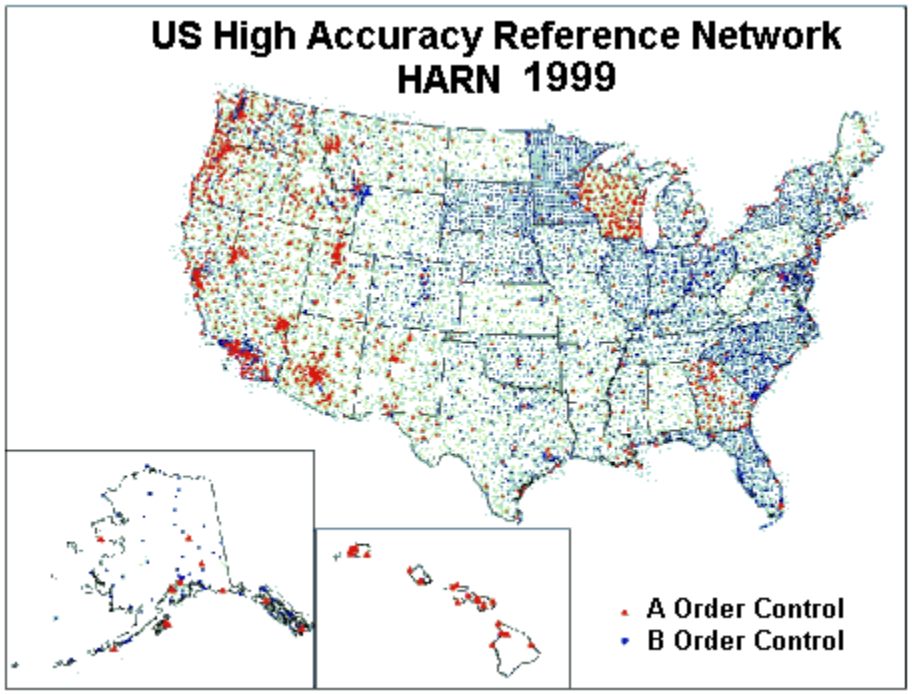
US Control Network
Before Satellite Surveying

First Order Triangulation
Second Order Triangulation
First Order Traverse

The US primary reference point is at the center of two of these legs. The points at the intersection of each set of lines are "monumented" and documented. Together these form the primary, or **first order network** for all coordinates in the US. The relative, point to point, nature of classical surveying is clear form this form of network.

C. Satellite Era Realizations

With the coming of **satellite surveying**, reference points could be far apart and essentially disjoint. They were still measured by relative surveying, on much higher accuracies could be obtained over much longer distances. This is clear from the current US primary realization. It looks like a set of points. The density is a function of the amount of work beyond the minimum that each state decided to perform.



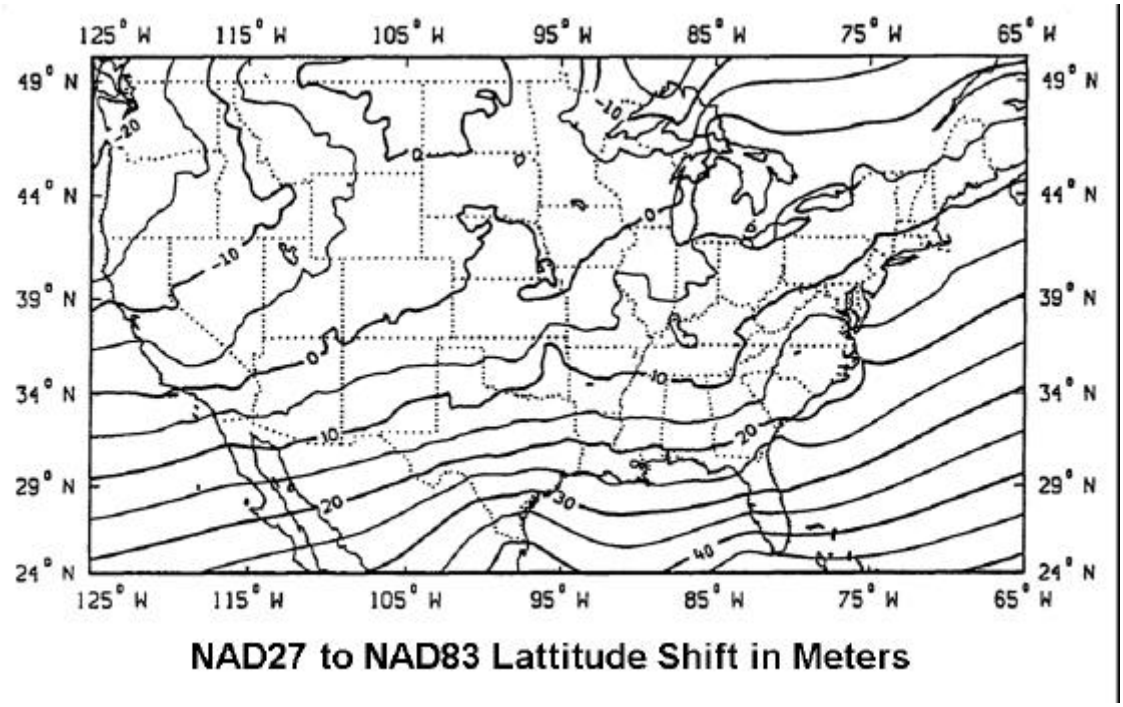
D. Datums As Rubber Sheets

Datums are established by taking a large number of high precision survey measurements at many points over the region of interest and solving for a best set of coordinates. In doing this there must be many more measurements than 2 times the number of points. The factor of 2 comes in because of the two horizontal coordinates at each location.

Even with a ratio of 10 to 1 of the number of measurements to unknown, the solutions can have significant errors. The **random errors** will be minimized, but **systematic errors** will remain. The surveys measurements were almost all relative measurements from point to point. Errors could accumulate. In addition only a small number of distances (called baselines) were commonly done due to the extreme expense. Any error in a base length would cause a scale error in all measurements dependent on it. This is an example of a systematic error.

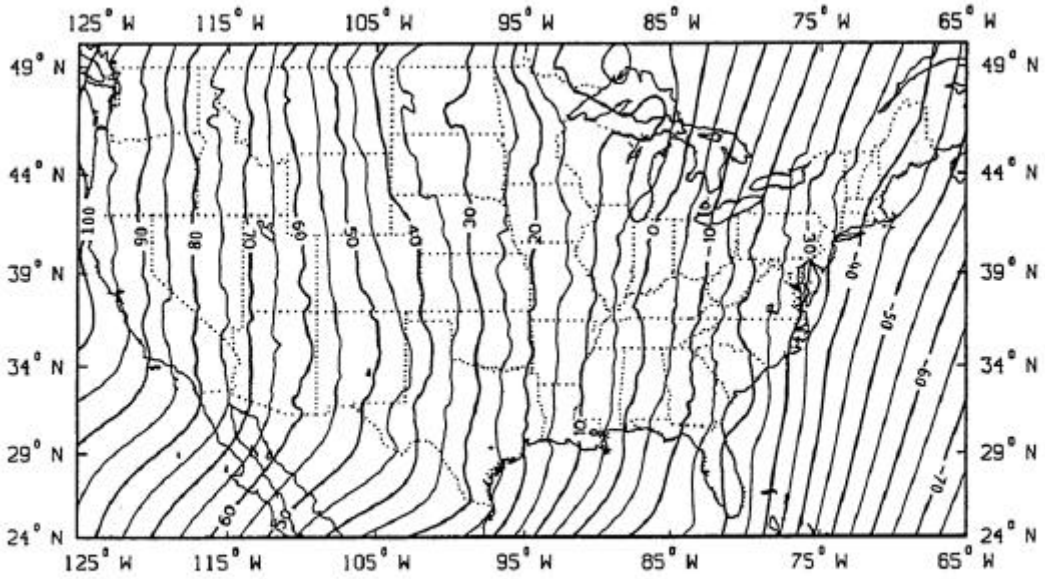
As a result there were often smoothly varying distortions that built up over distances. The resulting distortions often looked like a suspended sheet of rubber with weights at different points. This was very common in large area datums such as the North American Datum of 1927. Of course, the precise location and extent of the hills and valleys in this sheet were unknowable until a more accurate survey was done. For NAD27 this occurred with WGS72, the first extensive satellite based datum.

The North American Datum of 1983, NAD83, was the first large area civilian system based mainly on satellite surveying. This datum was significantly better the NAD27. The distortions in the old system show up in the **contour plots** of the differences between the latitude and longitude in the two datums. It is clear that the shifts are not constant, but vary <u>systematically</u> from place to place. The latitude shifts have a significant north-south gradient with bends in the west over the Rocky Mountains. Differences vary from -20 to +50 m.



NAD27 to NAD83 Lattitude Shift in Meters

The differences in the longitude have a predominate east-west gradient. The values vary from -40 m on the east coast to +100 m on the west coast. It is clear that over a small area map such as a 7.5' quad, the shift will be almost constant. But for the entire US there as significant, systematic variations.



NAD27 to NAD83 Longitude Shift in Meters

VI. What Datum Am I On

Because coordinates in different datums can differ by 100's of meters (or even a kilometer in the far east), it is important to know what datum you are on. There are usually two questions:

What datum is my map or **database** on?
What datum is my navigation system or survey on?

Clearly if the answer is the same, the map/data base and the position sensor can work together. Ships have gone aground when this was not so.

A. Map Datums - Paper and Electronic

The database of a map is usually listed in the **legend of the map**. In fact there are is usually both a horizontal and vertical datum listed. Today it is not uncommon to see two horizontal datums listed, on for the original map and one for some overprinting. This is how USGS has updated a lot of topographical maps from NAD27 to NAD83. But you have to read the legend carefully to notice this. In fact there are quite a few military maps issued by NIMA that use the same technique. The maps of the Balkans distributed in the mid 1990's were on the European Datum of 1950 with annotations in the legends on how to shift the positions to WGS 84.

For computer data files the issue is much more difficult. The data from the legends of the maps is usually preserved, but often not displayed. If the map is simply scanned as an image the legend is there, but the data is not usually "**registered**" or set up for computer reading of accurate coordinates. If the map has been entered into a **Geographical Information System (GIS)** it may well be registered, but in this case the legend data is present only in an auxiliary file. This type of legend data is called "**metadata**".

B. Navigation and Survey Equipment Datums

What datum is GPS on? The answer depends on how the GPS receiver generated the solution. **Stand alone GPS** and **Differential GPS (DGPS)** have different answers. And there are both a general answer, and a more precise answer.

1. Standalone GPS Users

For the stand alone user the simple answer is WGS 84. GPS operates by measuring ranges from the satellites to the user. In order to convert these ranges into positions, the locations of the satellites at the times the measurements were made is needed. Having a range and not knowing where it is from is not useful. The time history of the satellite positions is called the **ephemeris** of the satellites. <u>So a stand-alone user is on the datum of the ephemeris he uses.</u>

In the most common case the navigator or surveyor uses the ephemeris that is present on the signal broadcast by the satellites. This is called the **Broadcast Ephemeris (BCE)**. This information is "on WGS84". The quotes denote that this answer is not precisely correct, or is complicated by history.

There have been **Precise Ephemeris (PE's)** available for over 20 years. These are post-fit ephemeris based on a large set of ground stations. These are available from several civilian sources. There is even a consortium that produces a blended set of several PE's. In the past these were used for post fit work because it was a week or more after the fact that the PE's became available. After 2000, they have become available at very short delays. There even is a rapid prediction service that generates projected PE's that are much better than BCE out to a day or so. PE's are usually on the latest International Terrestrial Reference System - ITRF2000 in 2002. If you use these, you are on ITRF 2000.

The more complex answer for the BCE's puts these also on an ITRF. An ephemeris is computed from GPS observations made at known, fixed locations. The datum of an ephemeris is determined by the coordinates used for the antennas of these observations. These antenna locations for the **Operational Control System (OCS)** stations have been adjusted several times. This has effectively changed the datum of WGS84 as realized by the GSP BCE's.

In order to avoid confusion, the name of the datum was not changed when the adjustments were made. A suffix was added, but not widely used or know outside the geodesy community. The BCE datums were/are:

BCE Datum Name	Implemented	Matches
WGS 84	1980	Original WGS84
WGS 84 (G730)	1-2-1994	ITRF94
WGS 84 (G873)	9-27-1996	ITRF96
WGS 84 (G1150)	1- 2002	ITRF2000

GPS time is counted in weeks after January 1980. The number in the Gnnn is the GPS week number of the change. The first change moved coordinates about a meter. The last three changes have been much smaller. For the general navigator these changes may not be significant. There are some precision applications where these differences are important.

2. Differential GPS Users

With DGPS a reference station at a known location measures the errors in ranges to individual satellites and sends these to remote users over some communication link. In order to compute the errors, the reference station needs to know its position. The datum of the coordinates used for this position becomes the datum of all DGPS solutions that

use those corrections. If you use DGPS you are on the datum used by the reference station.

3. Other Electronic Navigation Systems

Other electronic navigation systems are much like DGPS. The coordinates used for the stations that transmit the signals or transpond signals define the datum. Often these are not accurate enough for meter level distinctions to be important. However in Asia, the difference between WGS 84 and other local datums can be large. The user must take care to know which datum his navigation system and maps are on.

4. Surveying

Surveying is much like DGPS. Both **classical surveying** and **satellite surveying** is usually done point to point. In a surveyed network there must be at least one known point. In large surveys there will be more. These know points are "held constant" in the analysis of the data. The datum used for the point(s) held constant defines the datum of the other surveyed points.

This is true of almost all GPS surveying as well as classical survey techniques. Data is taken at the same time with GPS receivers at different locations. The relative locations are then computed.

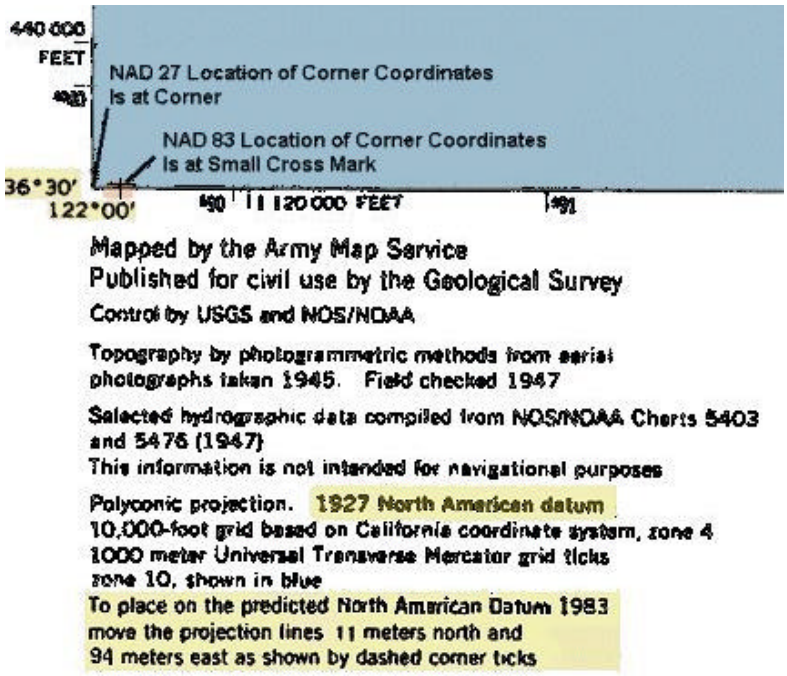
There is one exception, **Absolute GPS Surveying**. Large government agencies have to establish the primary points in remote areas. In this case a more complex post analysis is done on GPS data taken at a fixed location over several days. The analysis needs Precise Ephemeris to achieve survey quality positions. The answer is on the datum of the PE's used.

VII. Datum Transformations

A. Basic Methods

Having data in one datum and needing the coordinates in other is a common occurrence. It happens routinely when you use a GPS receiver outside of North America. It may be hidden from the user, but a transformation must occur to display coordinates from a GPS receiver in any other datum than WGS84. Of course, map and chart makers need to often make these transformations. Re-surveying everything each time a datum change is needed is impractical.

Over any small area, say 100 km or so, the transformation will be just a constant shift in latitude, longitude and height. This is a practical statement, and as accuracy requirements increase, the area over which a simple offset can be used gets smaller. This is one common way that maps are "updated". An offset is just printed in the legend.



This is an example of a US Geological Survey topographic map that has been updated from NAD27 to NAD83 in this way. The basic map is the old NAD27 map. You must read the legend and make an adjustment if you want NAD83 (the same as WGS84) coordinates. The location of the corner coordinates is also show as a small cross in each corner. This is very useful in deciding whether to add or subtract the adjustments.

There still is the issue of how to compute these offsets for each map.

There are three common methods of making these transformations from one datum to another. In the science world, the transformation is often viewed from a vector

perspective. The coordinates are transformed from Cartesian ECEF XYZ values of one datum to another. If latitude, longitude and height (LLH) are given or needed, the conversion to ECEF are done before the vector mathematics and then the new coordinates are converted back to LLH. Of course an ellipsoid definition (an "a" and "f") are needed for the LLH to/from ECEF. This method is usually called a **7-parameter transformation.** There are abridged versions of it with only **4 and 3 parameter transformations**.

A common method of directly transforming latitude, longitude, and height is the **Molodensky transformation**. This is a complex formula for the shift in latitude, longitude and height. It is complex because it is really a vector equation that is written out in its components. Also the values are scaled from lengths to arc-seconds of latitude and longitude. This method was very common before computers. It is still the most common method. There is a rudimentary method to deal with the distortion in datums. The number of parameters in this method is small.

A third method takes into account the distortions in the older transformations. A series of best-fit equations in the differences are generated. The equations are usually the same over a large area, such as the US, but there are different coefficients for small areas. This makes for a large database, but that is required if you wish an accurate transformation over a large area. This is often called the **Multiple Regression Method**.

B. Vector Method - The 7 Parameter Transform

The vector method is commonly used for the newer datums, or "frames". In this case it is assumed that there are negligible distortions and only some global changes are needed. The method deals in the earth centered, earth fixed, Cartesian coordinates (x,y,z).

It is assumed that there are three types of differences between the two frames:

a. The origin is different and a vector offset is given,
b. There is a rotation about each axis, and
c. There may be a scale change.

All these changes are assumed to be so small that many small parameter assumptions are valid. For example the sine of the rotations is replaced by the angle in radians and the cosine is replaced by 1. The order of the rotations is assumed un-important, something that is not true in general. The scale adjustment is also folded into the rotation matrix, also being order independent in this approximation. Usually the rotations are published in units of **milli-arc-seconds (mas)** or .001" . One mas is 5×10^{-9} radians. The **scale change** is also usually on the order of 10^{-9} .

The transformation equation commonly is found in two different notations. The older notations is:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{New} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{old} + \begin{bmatrix} \Delta x \\ \Delta y \\ \Delta z \end{bmatrix} + \begin{bmatrix} \Delta s & \omega & -\psi \\ -\omega & \Delta s & \varepsilon \\ \psi & -\varepsilon & \Delta s \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{old}$$

Here:

$(\Delta x, \Delta y, \Delta z)$ is the shift in the origin,
Δs is the scale value,
ω is a rotation angle in radians about the z axis,
ψ is a rotation angle in radians about the y axis, and
ε is a rotation angle in radians about the x axis.

In some cases the vector on the right of the matrix is written as $(x-x_0, y-y_0, z-z_0)$ where the zero values are the "primary point" of the original frame. In modern science work this is usually the center of the earth and thus omitted.

The alternate notation, common in the publications of information about the International Terrestrial Reference Frames (ITRF)'s is very similar in form, but uses different symbols.

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{New} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{old} + \begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} + \begin{bmatrix} s & \omega_z & -\omega_y \\ -\omega_z & s & \omega_x \\ \omega_y & -\omega_x & s \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}_{old}$$

This general vector transformation is often called a "7 parameter transformation". It is usually used for transformation between datums with low distortion, such as the WGS's and ITRF's, which cover the world or a large area. For transformations over smaller areas there are "**4-parameter transformations**" that drop the rotations and the very common "**3-parameter transformations**" that just have a vector offset. Over a small area, most datum transformation can be adequately represented by the 3-parameter version.

C. Molodensky - The Historically Common Method

1. Vector Viewpoint of Molodensky Transformation

The basis of the Molodensky transformation is to assume that all differences are due to:

a. A shift in origin by a vector $\vec{\Delta}$ with components $(\Delta X, \Delta Y, \Delta Z)$,
b. A difference in ellipsoids of size, Δa and flattening Δf, and

c. All changes are handled with one term in the Taylor series.

There are two additional things that complicate the equations. First the shift effects are written out in components, not using vector notation. Second the results, which are initially a shift in east, north and up in distance units are converted to angles for latitude and longitude. This involves the **radius of curvature** in the two directions. (See separate section on geodetic coordinate conversions for details of $\mathbf{R_N}$ **and** $\mathbf{R_M}$.)

To begin the equations, note that the **unit vectors in the east, north, and up** directions are given by:

$$\begin{aligned} \hat{e}_E &= (-\sin\lambda, \cos\lambda, 0) \\ \hat{e}_N &= (-\sin\phi\cos\lambda, -\sin\phi\sin\lambda, \cos\phi) \\ \hat{e}_U &= (\cos\phi\cos\lambda, \cos\phi\sin\lambda, \sin\phi) \end{aligned}$$

The effects of the origin shift $\vec{\Delta}$ are easily obtained by taking the **dot product** of the unit vectors with $\vec{\Delta}$. The effects of the ellipsoid difference can be obtained with a few derivatives. This gives, in distance units for the shift in East, North, and Up:

$$\begin{aligned} \Delta E &= \vec{e}_E \bullet \vec{\Delta} \\ \Delta N &= \vec{e}_N \bullet \vec{\Delta} + \frac{\Delta a}{a} e^2 R_N \cos\phi\sin\phi + \Delta f\left[\frac{a}{b}R_M + \frac{b}{a}R_N\right]\cos\phi\sin\phi \\ \Delta H &= \hat{e}_U \bullet \vec{\Delta} - \Delta a\frac{a}{R_N} + \Delta f\frac{b}{a}R_N\sin^2\phi \end{aligned}$$

The equation for height is left as show above. The usual procedure for the East-Change is to convert it to arc-seconds of longitude and to convert North-Change to arc-seconds of latitude. This is done with the correct radii of curvature and the sine of one arc-second. The scaling lengths are:

$$\begin{aligned} L_E &= (R_N + H)\cos\phi\sin 1'' \\ L_N &= (R_M + H)\sin 1'' \end{aligned}$$

The usual equations are found by dividing by these values:

$$\begin{aligned} \Delta\lambda'' &= \frac{\Delta E}{L_E} \\ \Delta\phi'' &= \frac{\Delta N}{L_N} \end{aligned}$$

2. The Usual Statement of the Molodensky Transformation

Older textbooks and manuals give equations for the Molodensky transformation that would be used in a hand calculation. These give directly the shifts in latitude, longitude, and height. The angular values are in arcseconds. These formulas will be given here for completeness.

The standard form is often given as:

$$\begin{aligned}
\Delta\phi'' &= \{-\Delta X \sin\phi\cos\lambda - \Delta Y \sin\phi\sin\lambda + \Delta Z\cos\phi + \\
&\quad \frac{\Delta a}{a}(R_N e^2 \sin\phi\cos\phi) + \Delta f\left[\frac{a}{b}R_M + \frac{b}{a}R_N\right]\sin\varphi\cos\phi\Big\} \Big/ [(R_M + h)\sin 1'']\\
\Delta\lambda'' &= \{-\Delta X \sin\lambda + \Delta Y\cos\lambda\}/[(R_N + h)\cos\phi\sin 1'']\\
\Delta h &= \Delta X\cos\phi\cos\lambda + \Delta Y\cos\phi\sin\lambda + \Delta Z\sin\phi - \Delta a\frac{a}{R_N} + \Delta f\ b\frac{R_N}{a}\sin^2\phi
\end{aligned}$$

This is often shortened to the Abridged Molodensky Transform given by:

$$\begin{aligned}
\Delta\phi'' &= \{-\Delta X \sin\phi\cos\lambda - \Delta Y \sin\phi\sin\lambda + \Delta Z\cos\phi + \\
&\quad (a\,\Delta f + \Delta a\, f)\sin 2\varphi\}/[R_M \sin 1'']\\
\Delta\lambda'' &= \{-\Delta X \sin\lambda + \Delta Y\cos\lambda\}/[R_N \cos\phi\sin 1'']\\
\Delta h &= \Delta X\cos\phi\cos\lambda + \Delta Y\cos\phi\sin\lambda + \Delta Z\sin\phi - \Delta a + (a\Delta f + f\Delta a)\sin^2\phi
\end{aligned}$$

The abridged form is found by dropping any terms that are second order in small parameters (f, e, etc.). The addition of h to the radii is ignored, as the two are usually different by a factor of 1000 or more. For this reason, it is not critical if the height used is **orthometric** (H) or **geodetic** (h) where added to the R's. (See section on heights and **geoid** for details.) It will make only a small difference in an already small correction.

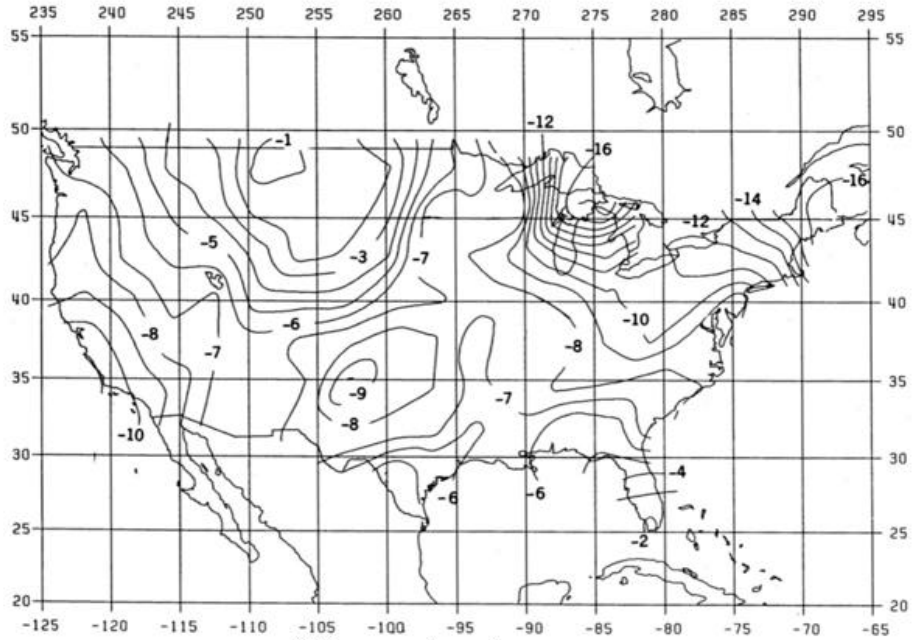
3. Modern Uses of Molodensky Transformation

The implementation of the Molodensky transform in imbedded in many geodetic programs. When WGS84 was initially published, a set of Molodensky transformation parameters was published between WGS84 and about 50 other datums. These were incorporated into most GPS receivers of the time, including all US Defense Department receivers. That is still the case.

The problem with the Molodensky transformation is the limited amount of data that it uses. There are only 5 numbers, three in ÄX,ÄY, ÄZ and two in Äa and Äf. For small areas this is fine. But for larger areas such as the US with NAD27 and Europe with

EU50, significant errors (10's of meters) resulted from the use of a single set of parameters. The original datums were distorted and needed more data to provide good datum shifts over the area of use. The ad hoc solution was to allow ÄX,ÄY, ÄZ to be functions of position. These were then free parameters that were determined when the new datum was generated. An example of the resulting "**datum shift parameter ÄX** " is shown below. There are similar maps for ÄY **and ÄZ**. In fact NIMA has published maps of ÄX,ÄY, and ÄZ for all major large area datums it uses.

The map for the "best" ÄX value is shown below. There are similar maps for ÄY and ÄZ. The ÄX values ranges from -16 to -1 m, the ÄY values from 172 to 187 m and ÄZ from 157 to 165m. There is about a 10 to 15 m variation in each axis. Of course the best-fit ellipsoids have only one offset, but in order to get better results from the Molodensky transformation, the value of $\vec{\Delta}$ is allowed to vary.

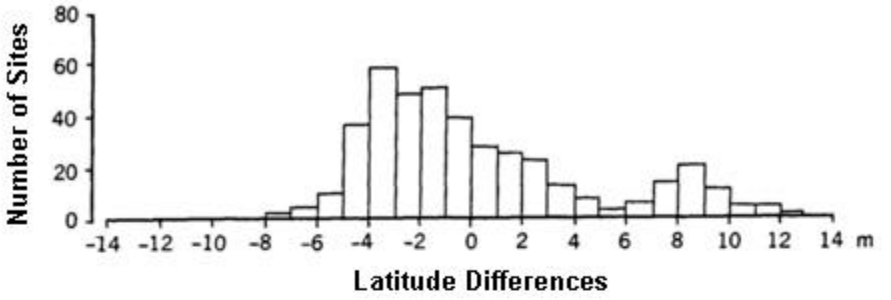


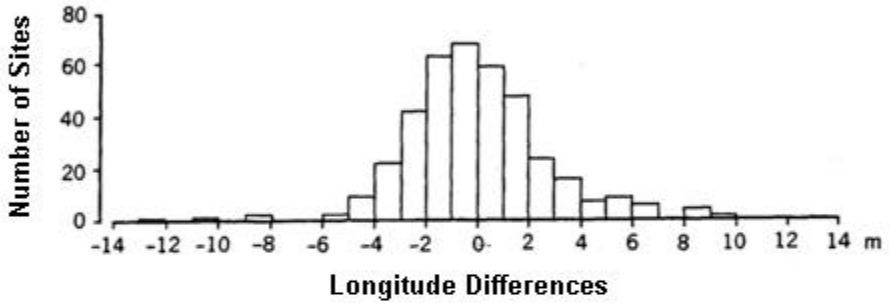
ΔX Component - meters
Datum Shift NAD 27 to WGS 84

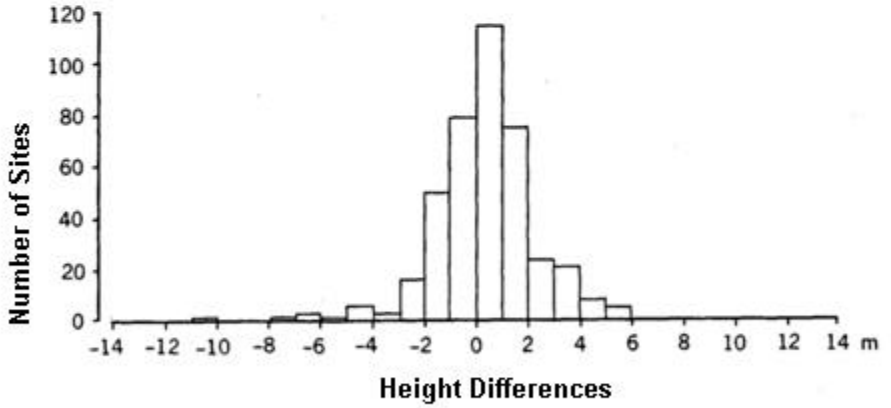
These maps are seldom used in practice. The computer programs that do datum transformations use only a few Ä's. For the continental US it is common to use just two sets, on for east of the Mississippi river and one if west of it. This is true of the conversion routines in GPS receivers as well as the DoD supplied PC conversion programs MADTRAN and **GEOTRANS**.

In the original unclassified report on the WGS84 datum, a table of about 50 sets of Ä's was given. There were only three sets for the continental US, the entire US, the US east of the Mississippi, and the US west of the Mississippi. This table has been incorporated into GPS receivers and many non-scientific computer programs.

When the WGS84 datum was developed, there were many points accurately surveyed with satellite methods that were already known in NAD27 coordinates. The "datum shift parameters" for the whole US were tested with this set of points. The following is from the DMA WGS84 Report supplement.







Measured Differences
Transformed NAD27 vs. Satellite Measurements
Continent Average ΔX, ΔY, ΔZ

It is clear that for accuracies of 20 meters, this is adequate. However at 5 meters it is not. In particular the latitude errors are not random and have significant systematic character.

D. Local Fit Equations - Multiple Regression Method

The **multiple regression equations (MRE)** are ad hoc equations that provide for the shift in latitude and longitude as a function of position. They take the form:

$$\Delta\phi'' = A_0 + A_{1,0}U + A_{0,1}V + A_{2,0}U^2 + A_{1,1}UV + A_{0,2}V^2 + \ldots$$

and a similar equation for $\Delta\lambda''$ using another set of coefficients $B_{i,j}$. All the information is in the coefficients. The values of the independent variables, U and V are scaled latitude and longitude,

$$\begin{aligned} U &= k(\phi - \phi_m) \\ V &= k(\lambda - \lambda_m) \end{aligned}$$

with k being a constant and (ϕ_m, λ_m) being a point near the middle of the area of validity.

NIMA published MRE shifts for the NAD27 to WGS84 valid for the entire US. These are shown in the figure below.

Multiple Regression Equations (MREs)
for Transforming
North American Datum 1927 (NAS) to WGS 84

Area of Applicability : **USA (Continental contiguous land areas only; excluding Alaska and Islands)**

MRE coefficients for ϕ and λ are :

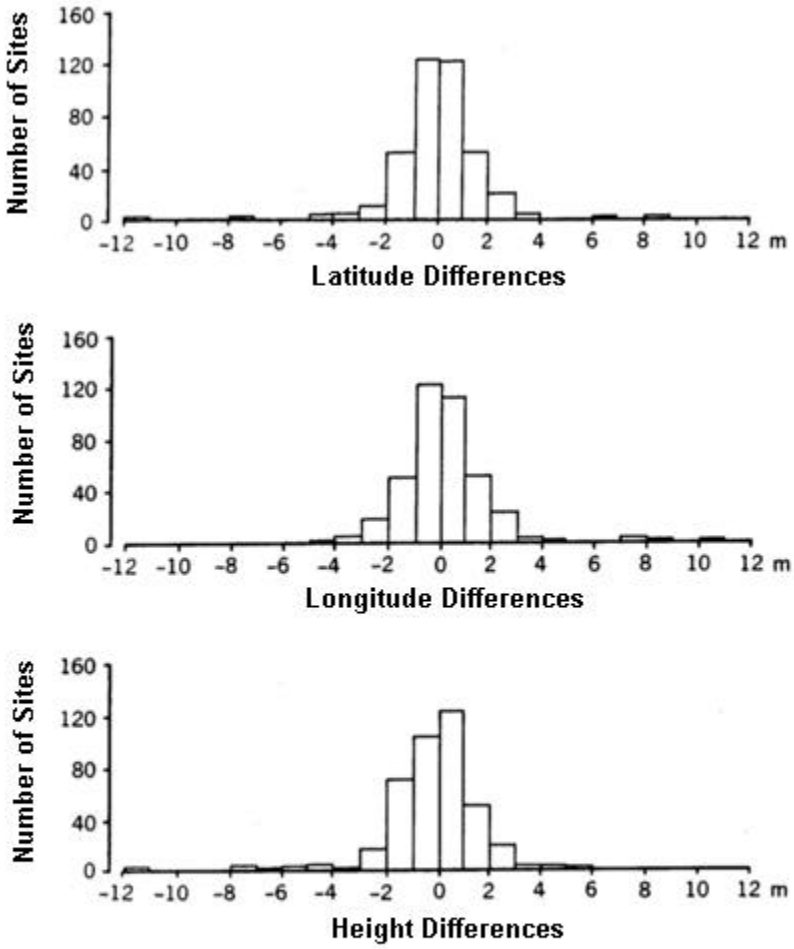
$$\begin{aligned} \Delta\phi'' = {} & 0.16984 - 0.76173\,U + 0.09585\,V + 1.09919\,U^2 - 4.57801\,U^3 - 1.13239\,U^2V \\ & + 0.49831\,V^3 - 0.98399\,U^3V + 0.12415\,UV^3 + 0.11450\,V^4 + 27.05396\,U^5 \\ & + 2.03449\,U^4V + 0.73357\,U^2V^3 - 0.37548\,V^5 - 0.14197\,V^6 - 59.96555\,U^7 \\ & + 0.07439\,V^7 - 4.76082\,U^8 + 0.03385\,V^8 + 49.04320\,U^9 - 1.30575\,U^6V^3 \\ & - 0.07653\,U^3V^9 + 0.08646\,U^4V^9 \end{aligned}$$

$$\begin{aligned} \Delta\lambda'' = {} & -0.88437 + 2.05061\,V + 0.26361\,U^2 - 0.76804\,UV + 0.13374\,V^2 - 1.31974\,U^3 \\ & - 0.52162\,U^2V - 1.05853\,UV^2 - 0.49211\,U^2V^2 + 2.17204\,UV^3 - 0.06004\,V^4 \\ & + 0.30139\,U^4V + 1.88585\,UV^4 - 0.81162\,UV^5 - 0.05183\,V^6 - 0.96723\,UV^6 \\ & - 0.12948\,U^3V^5 + 3.41827\,U^9 - 0.44507\,U^8V + 0.18882\,UV^8 - 0.01444\,V^9 \\ & + 0.04794\,UV^9 - 0.59013\,U^9V^3 \end{aligned}$$

Where : $U = K(\phi - 37°)$; $V = K(\lambda + 95°)$; $K = 0.05235988$

NOTE : Input ϕ as (-) from 90°S to 0°N in degrees.

While there are about two dozen coefficients in these expressions, these only do a fair job in representing the shifts. A set of a few hundred geodetic marks with NAD27 coordinates and accurate satellite positions was used to test this shift. The results were good but there were areas with significant distortions. A distribution of the results was computed.



Measured Differences
Transformed NAD27 vs. Satellite Measurments
All Continent Multiple Regression Equations

In order to do a better job, more data is needed. The approach used by the NGS for North America and a few other nations is to take the massive data sets used to define the newer, better datum and solve for simple fit equations in terms of difference of latitude and longitude from some reference points. And then have many of these reference points and apply the fit from it only in a small area. This involves a large data set. The NGS computer program NADCON uses this technique. It reads a database of

about a megabyte. (Of course "large database" is a relative scale. Today this is not considered very large.)