Vectors

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I.   Basic Definitions

Vectors are common entities in our world. In physic, engineering and mathematics they are more defined in specific ways. This note will try and explain the concepts and notation used in these fields. The term vector often has a slightly different meaning in computer work. This overlaps with the physics definition at times.[*]

There is a distinction between the basic entity called a vector, and the representations we use to manipulate it. Many types of things are vectors. They exist independent of our particular representation.   <u>A vector is usually defined as an entity that has both magnitude and direction.</u> By contrast, a thing that has just a magnitude is called a <u>scalar</u>. The mass of a baseball is a scalar. The velocity as it comes off the bat is a vector. The magnitude of the velocity is called the speed.

Note that mass, velocity, and speed all have units. The baseball may be 145 grams. It diameter might be 7.3 cm. And it's speed off the bat 80 mph or 115 ft/s or 35 m/s. (This gives a 400 ft or 120 m maximum horizontal distance ignoring air resistance.) The same physical quantity has different numerical values when you change units. There are entities called numbers. These are unitless. They are the basis of expressing values when we add units. In the same way there are unit vector, which will be described below.

While the speed might be 35 m/s, the velocity might be 35 m/s in the direction 45 deg up and 35 degrees to the right of the line to the pitcher (right field home run).   This is a cumbersome description. A more convenient description uses a coordinate system. This will be discussed below.

There are many examples of vectors in our world. These include the position of an object, the velocity of a car, the velocity of the wind or water (a current), the acceleration you experience when you step off a diving board, the force that the earth's gravity field has on you that makes you fall toward the water etc.   So position, rates of change of position (velocity, acceleration), and forces are vectors. (There are some other important vectors in physics, such as momentum and angular momentum, but we will leave that to a discussion on physics.)

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[*] In computer work a "vector" is usually just a set of numbers stored in sequential areas of memory. The computer vector can be referred to by its basic name to mean the entire set or with a subscript for an individual number.

II.   Vector Examples

The basic picture of a vector is shown in figure 1. Usually represented as a line with an arrow at one end that defines the direction.   This is called the head end. The other end is the tail. A vector is often written as symbol with a small arrow (vector sign) over it as $\vec{r}$. It sometime is denoted by a bold symbol **r** . The vector symbol notation will be used here.


$\vec{r}$
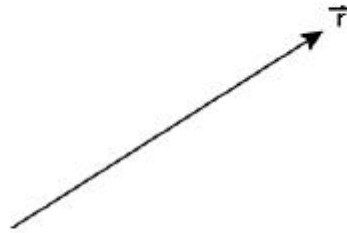
Vectors and Representations

The vector is thought of as being located at the tail position. There might just be one vector that represents something that does not change with time. For example the force of gravity in a small area such as a baseball stadium is constant in time and almost independent of location. Or it might change such as the position of a hit ball. Note that the velocity of the ball also changes during its flight. It starts going up but ends coming down. The direction has changed. So a vector might have a constant magnitude (say speed) but change because its direction changes. A car going around a curve at constant speed is a common example of this.

There are important cases where vectors exist at each point in some area or volume. For example the wind velocity at one point in time will vary with position. This is sometimes presented as a set of vector symbols on a weather chart. Currents near shore in the ocean also vary with position. We call these a <u>field of vectors</u>.

III. Vector Representations

A. Cartesian Vectors

The most common way to make it easier to describe or represent a vector is to use a coordinate system. This is usually a Cartesian system with two or three axis at right angles. Here, in figure 2, we will use only two dimensions for our examples.
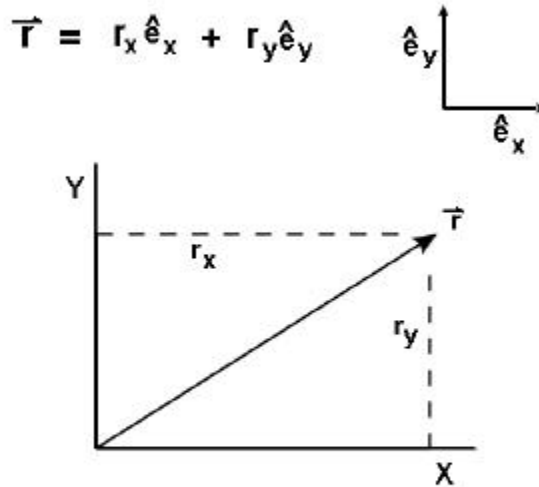
Vectors and Representations
Cartesian Components (r_x , r_y)

A set of axis, usually called X and Y, are laid down at right angles. If the origin is moved to the tail (location) of the vector, it is easy to find the Cartesian components. The vector $\vec{r}$ is said to have components in the X and Y direction. These are usually denoted either by a subscript, as in r_x and r_y for the X and Y components. They also are often just written as a set of numbers in parthenthis or brackets, (r_x, r_y). Notice that the order is important. The value (1,0) is on the X axis but (0,1) is on the Y axis.

This representation is really shorthand for:

$$\vec{r} = r_x \hat{e}_x + r_y \hat{e}_y$$

Here the values of the vector, with their units, are in the r_x and r_y . The new items are the <u>unit vectors</u> $\hat{e}_x, \hat{e}_y, \ and \ \hat{e}_z$. (Note in some physics books these are called **i, j**, and **k**). Unit vectors are the equivalent vector entities for the number 1. While the number 1 is the basis for scalars,

the unit vectors are the basis for vectors. There are an infinite number of unit vectors just as there are an infinite number of directions. But in two dimensions there are only two independent unit vectors. Given any two that are not the same (pointing in the same direction) we can represent all the others as a linear combination of these two. In three dimensions there are three independent unit vectors. It is useful to choose these to be perpendicular to each other. That is, they meet at right angles.

When we list the two or three component of a vector, we are implicitly using a set of unit vectors. This can be important and confusing in cases where two systems are commonly used. In the case of positioning on the earth, there are two common cases. For world wide studies, the coordinates often have the origin at the center of the earth, with the z axis along the polar or spin axis X going out the equator along the 0 longitude line and Y along the 90 East longitude line (under India). A second coordinate basis commonly used is East, North, Up. We do this implicitly when we give directions.

One important thing to note is that the (East, North, Up) system varies in direction depending on location. Just think of what direction Up represents in California vs. in England. The Up unit vectors form a field of vectors. The same is true of North and East.
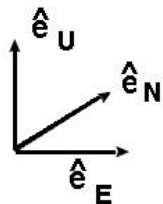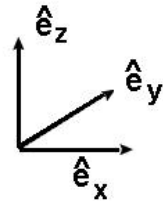
B.   Right Hand Systems

It is usual to have Cartesian coordinate systems be "right handed". A <u>right hand coordinate system</u> (RHS) is defined by the order of the coordinates. If you take the first coordinate and rotate it into the second, a screw would move in the direction of the third. Or looking down the third axis, rotating the first axis into the left represents a counter clockwise motion. It is called right handed because you can take a right hand and put the palm along the first axis, the fingers curl to the second and the thumb is along the third.
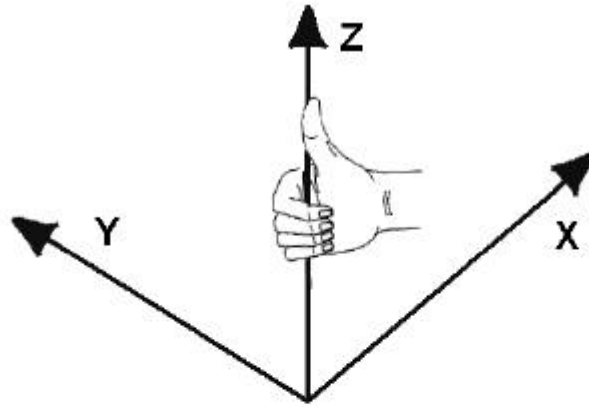
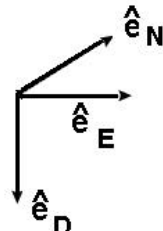**Right Hand Coordinate System**

When Right Hand Fingers Curl X to Y Thumb Points Along Positive Z Axis

The following figure has 4 examples of right hand coordinate systems.
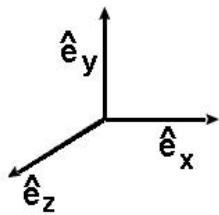

(East, North, Up)

(North, East, Down)

Right Hand Coordinate Systems

Notice that the top two can be changed into each other through rotations. This is not true if you interchange two axes, say X and Y in the upper left. If you do that you have a left hand coordinate system. Notice that the order of the east, north and up in a system is important. (East, North, Up) is a right hand system while (North, East, Up) is not. But if you use down instead of up, (North, East, Down) you again get a RHS. Satellite work is sometimes done in (E,N,U) and geophysical work in (N,E,D), both right hand systems.

C. Polar Coordinates

In two dimensions, it is sometimes useful to represent a vector by its' magnitude and direction. In this case only one number is needed to specify the direction. The case is shown in figure 4.
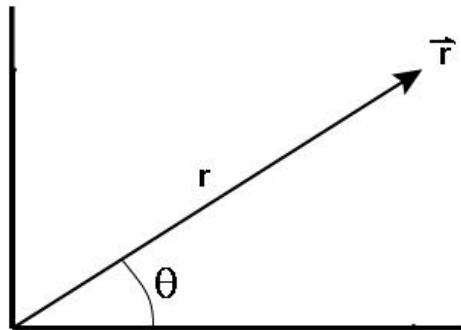
Vectors and Representations
Polar Representation
Magnitude r, angle θ

IV.   Vector Mathematics   Addition, Subtraction, Scaling

Just as you can add, subtract and multiply scalars (numbers), you can do the same operations on vectors. The multiplication between two vectors comes in two forms. These will be covered in the next two sections.

The easiest mathematical operation on vectors to understand is the multiplication by a scalar. If you multiple the vector $\vec{A}$ by the number two, you get another vector in the same direction with twice the length. Thus if s is a scalar and,

$$\vec{B} = s\,\vec{A}$$

Then the components of $\vec{B}$ are s (A_x, A_y, A_z) = (sA_x, sA_y, sA_z).

To add vectors using a graphical method, take place the tail of one at the head of the second. The vector from beginning to end of this combination is the sum. It does not make any difference which one you use as the first and second in this procedure. (Order will be important in other vector operations!).

An Example is given in the following figures. Here the vector $\vec{A}$ has components (10, 6) and the vector $\vec{B}$ component are (5, 1).

Add Vectors $\vec{A}$ and $\vec{B}$
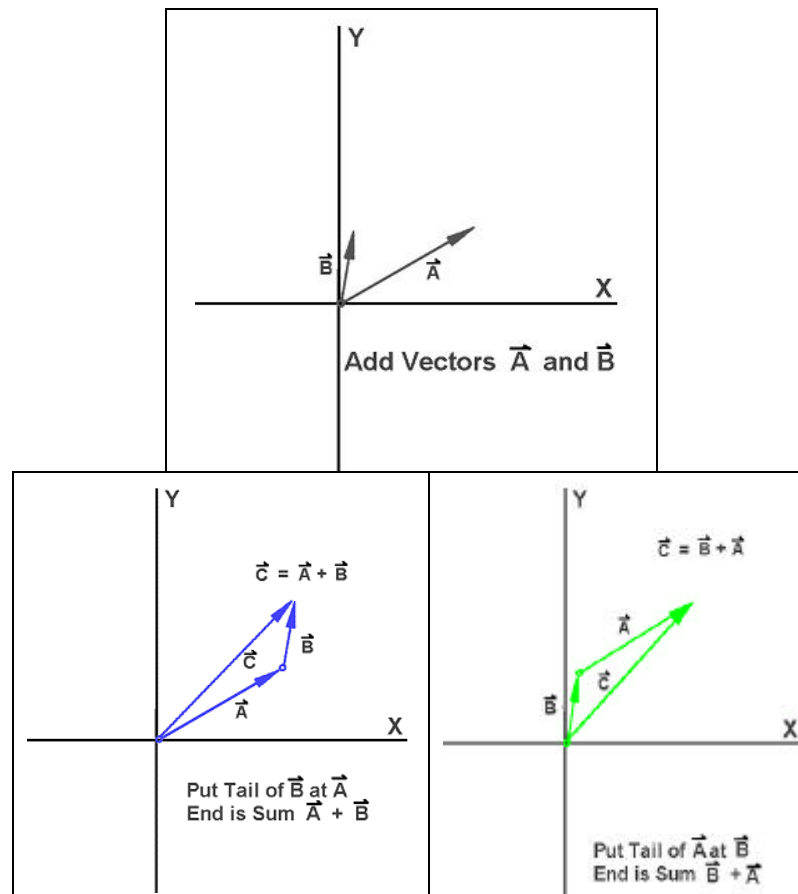
In order to obtain the coordinates of the sum, $\vec{C}$, they must be measured from the graph.

In the component method, one just adds vectors by adding the components. Thus

$$(C_x, C_y) = (A_x, A_y) + (B_x, B_y)$$

or

$$(15, 1) = (10, 6) + (5, 1).$$

This is much computationally much easier.

V. Inner or Dot Product

There are two ways to multiply two vectors together. The <u>inner product or dot product</u> takes two vectors and produces a scalar.

A. Basic Definition

The basic definition of the inner product of two vectors $\vec{A}$ and $\vec{B}$ is the magnitude (length) of each times the cosine of the angle between them.

$$\vec{A} \cdot \vec{B} = AB\cos\theta$$

Inner Vector Product
Dot Product
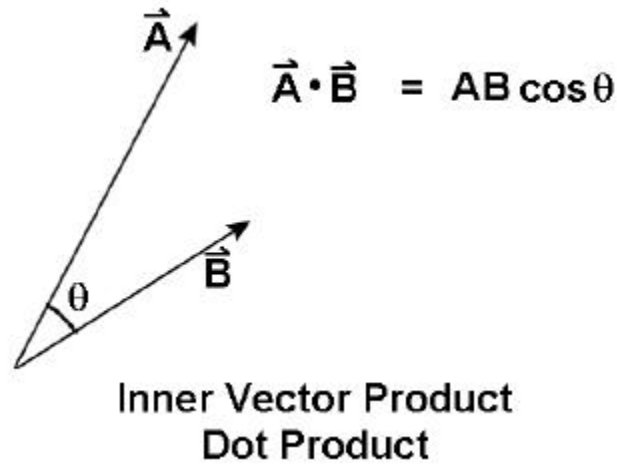
B. Computation Using Components

In order to find a practical method for computing the dot product, consider the vector components.

$$\vec{A} = (A_x, A_y, A_z)$$
$$\vec{B} = (B_x, B_y, B_z,)$$

But we can rewrite the vectors in terms of the components.

$$\vec{A} \bullet \vec{B} \;=\; \left[A_x\hat{e}_x + A_y\hat{e}_y + A_z\hat{e}_z\right]\bullet\left[B_x\hat{e}_x + B_y\hat{e}_y + B_z\hat{e}_z\right]$$
$$=\; [A_xB_x\hat{e}_x \bullet \hat{e}_x + A_xB_y\hat{e}_x \bullet \hat{e}_y + A_xB_z\hat{e}_x \bullet \hat{e}_z] +$$
$$[A_yB_x\hat{e}_y \bullet \hat{e}_x + ...]$$

Now the only vector quantities are the unit vectors. And because they are perpendicular to each other (that is orthogonal) the cosine factor is zero in a dot product between different unit vectors and one for the same one. That is

$$\hat{e}_i \bullet \hat{e}_i = \begin{cases} 1 & if \quad i = j \\ 0 & if \quad i \neq j \end{cases}$$

Thus we have only 3 of the nine terms non-zero. Thus the equation for the inner product in terms of the components along an orthogonal set of unit vectors is:

$$\vec{A} \bullet \vec{B} \;=\; A_xB_x + A_yB_y + A_zB_z$$

This involve only simple multiplication of scalars, something a computer does very easily. We do not have to compute any angles.

C. Vector Length

The length of a vector is the square root of the inner product with itself. Thus

$$\left|\vec{A}\right| \;=\; \sqrt{\vec{A} \bullet \vec{A}},$$

where the vertical bars are used to indicate the magnitude or length of the vector.

The graphical method of obtaining the length is to measure it. Using components is much more accurate and efficient. Using the last equation of the previous section:

$$\left|\vec{A}\right| \;=\; \sqrt{A_x^2 + A_y^2 + A_z^2}$$

D. Inner Product as a Projection Operator

The projection of one line on another is the length of the first line in the direction of the second. This can be found from simple geometry by forming a right triangle. This is identical to the procedure for the dot product.
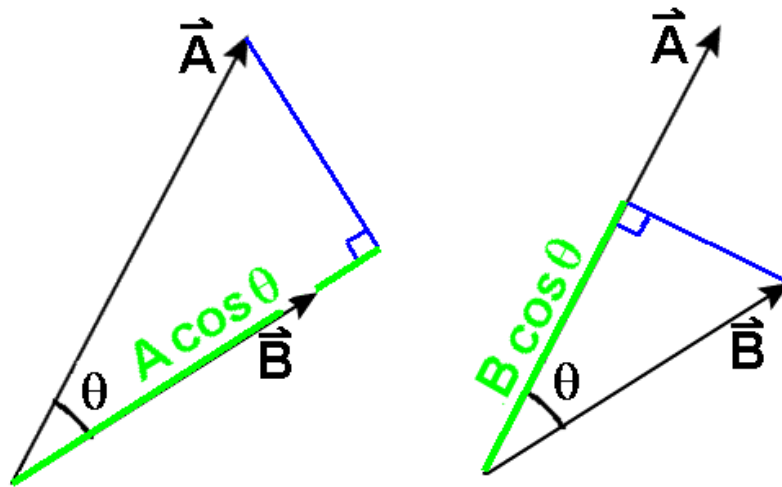
$$\vec{A} \cdot \vec{B} \; = \; AB \cos \theta$$

Inner Vector Product
Dot Product

If one of the vectors is of unit length, then the dot product just finds the component along that direction. That is we have a method of finding the components (A_x, A_y, A_z).

$$A_x \;\; = \;\; \vec{A} \bullet \hat{e}_x$$
$$A_y \;\; = \;\; \vec{A} \bullet \hat{e}_y$$
$$A_z \;\; = \;\; \vec{A} \bullet \hat{e}_z$$

VI.   Cross Product or Outer Product

A. Basic Definition

There is a multiplication of vectors that results in another "vector" (a pseudo vector as we shall see).   This is called the cross product. The cross product between two vectors is a vector perpendicular to the plane formed by the two vectors with a magnitude of the product of the magnitude of the two input vectors times the sine of the angle from the first to the second vector.

$$\left|\vec{A}\times\vec{B}\right| \;=\; \left|\vec{A}\right|\left|\vec{B}\right|\sin\theta$$

The sign of the direction is determined by the right hand rule. Rotating the first into the second it takes the direction a screw would take.   This means that

$$\vec{A}\times\vec{B} \;=\; -\vec{B}\times\vec{A}$$

and the order is important. This is not the case for the dot produce which is order independent.
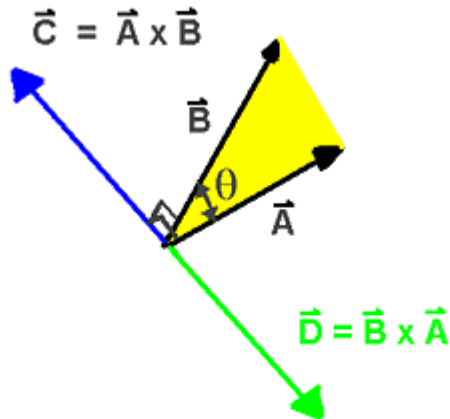
Cross Product
Direction by Right Hand Rule
Magnitude |AxB| = |A||B|sin θ

B. Cross Products Using Components

Because the unit vectors are perpendicular, the angle will be 90 degrees and the sine factor will be 1.   Thus the output of the cross of one of the basis unit vectors into a second is a third.

$$\hat{e}_x \times \hat{e}_y \;=\; \hat{e}_z$$

$$\hat{e}_y \times \hat{e}_z \;=\; \hat{e}_x$$

$$\hat{e}_z \times \hat{e}_x \;=\; \hat{e}_y$$

If you take the cross product in a cyclic manor (x and y or y and z or z and y) the sign of the result is positive. If you invert the order the sign is minus. The cross product of one of these unit vectors with itself is zero because the angle is zero.

The component form of the cross product is therefore:

$$\vec{C} \;=\; \vec{A} \times \vec{B}$$
$$\;=\; (A_y B_z - A_z B_y, \quad A_y B_z - A_z B_y, \quad A_y B_z - A_z B_y)$$

C. Cross Product as Pseudo Vector

The result of a cross product between two vector acts almost like a vector. The only exception is if the axes are inverted. This is called mirror symmetry. In this case you use a new set of unit vectors along the negative of each axis. This will multiply each component of all vectors by minus 1. Under this operation the component are inverted.

But in the cross product the product of two of these minus ones will be plus one. Thus the vector that comes out of a cross product is not inverted. For this reason these vectors are called pseudo-vectors.

In fact the situation is slightly more complicated. The result of going from a right hand coordinate system to a left hand coordinate system does not work for a cross product. So more than the mirror symmetry is a problem. This is the main reason that we use only RHS by convention. (We could have chosen to use only left handed systems.) Thus you can use cross products as a vector if you are careful about generating special rules of handling. These are the formulas of cross products with the additional not that you should not change the parity of your basis.

D. Cross Product Examples

Why all the trouble with cross products. Because they occur when dealing with rotational effects. Angular momentum is the result of a cross product of the position and momentum vectors. The spin vector, a vector along the spin axis and with magnitude of the rotation rate is a pseudo-vector. In going from an inertial to a rotating coordinated system (such as any

coordinates fixed to the earth) terms arise that are pseudo vectors. The Corollas effect is one example.

Cross products also occur often in the equations of electromagnetic fields. This is because there are really not independent electric and magnetic field, but one combined field that is correctly represented by a tensor of dimension 2. Again we use the simpler vector formalism and some ad hoc rules that are just the rules of cross products.
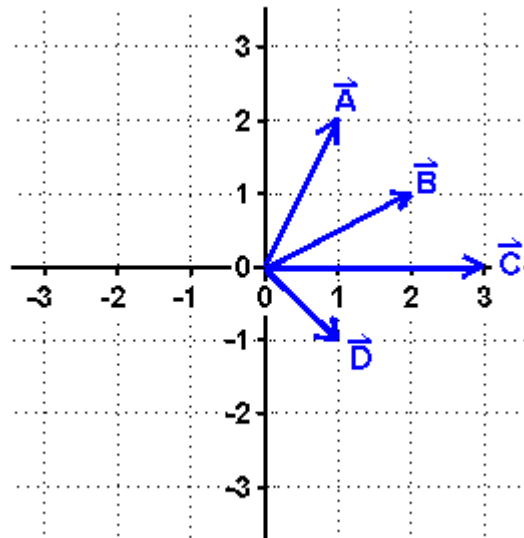
VII.   Some Examples

In the following diagram we have 4 vectors with components:

$$\vec{A} = (1,2)$$
$$\vec{B} = (2,1)$$
$$\vec{C} = (3,0)$$
$$\vec{D} = (1,-1)$$

From the vector diagram one can see that subtracting D form C yields B. Going another step you can subtract D from B and get A. These relationships can easily be verified from the components. In many cases it is easier to find relationships from components than a vector diagram. However the diagram may lead to into what is going on.

If a boat to wishes to cross a river that is in the X direction, it must go in the Y direction. But if D is the current, then the boat needs a vector of B relative to the water to go straight across the river. (C = B + D). Of course you would use components to find numeric values.

The inner or dot product of A and B is easily found from components to be:

$$\vec{A} \bullet \vec{B} \;=\; 2*1+1*2 \;=\; 4$$
$$=\; 1*2+2*1$$
$$=\; \vec{B} \bullet \vec{A}$$

The cross product is not defined for two-dimensional vectors, only those in three dimensions. If we assume a Z component of zero for these vectors then if

$$\vec{E} \;=\; \vec{A} \times \vec{D}$$

the vector E will be in the Z direction. (The X and Y components will be zero.) That is

$$\vec{A} = (1,2,0)$$
$$\vec{D} = (1,-1,0)$$
$$\vec{E} \;=\; \vec{A} \times \vec{D}$$

The component of E are found from:

$$E_x \;=\; [2*0 - 0*(-1)] \;=\; 0$$
$$E_y \;=\; [0*1 - 1*0 \;\;] \;=\; 0$$
$$E_z \;=\; [\,1*(-1) - 2*1] \;=\; -3$$

Taking the cross product in the other order will change the sign of the answer.

VIII. Summary of Vector Properties

Basics Quantities - Scalars and Vectors
There are quantities, such as velocity, that have both magnitude and direction. These are vectors. A quantity with only magnitude is a scalar.

Nomenclature

$\vec{A}, \vec{B}, \vec{C}, \vec{D}$ will be vectors

$|\vec{A}|$ is the magnitude or length of the vector $\vec{A}$. It is a scalar.

$\hat{e}_x, \hat{e}_y, \hat{e}_z$ will be a set of orthogonal unit vectors along the x,y,z axes

     (Some authors use **i,j,k** for these unit vectors.)

A_x, A_y, A_z are scalars that are the components of the vector $\vec{A}$

s   is a scalar

Component Notation

$$\begin{aligned}\vec{A} &= A_x\hat{e}_x + A_y\hat{e}_y + A_z\hat{e}_z \\ &= (A_x, A_y, A_z)\end{aligned}$$

Scalar Multiplication

$$s\,\vec{A} = s\,(A_x, A_y, A_z) = (sA_x, sA_y, sA_z)$$
$$\begin{aligned}s\,(\vec{A} + \vec{B}) &= s\,\vec{A} + s\,\vec{B} \\ &= (sA_x + sB_x, sA_y + sB_y, sA_z + sB_z)\end{aligned}$$

Addition

$$\begin{aligned}\vec{A} + \vec{B} &= \vec{B} + \vec{A} \\ &= (A_x + B_x, A_y + B_y, A_z + B_z)\end{aligned}$$

Dot Product or Inner Product

$\vec{A} \bullet \vec{B}$ is a scalar.
$$\begin{aligned}\vec{A} \bullet \vec{B} &= |\vec{A}|\,|\vec{B}|\cos\theta \\ &= A_xB_x + A_yB_y + A_zB_z\end{aligned}$$
Where θ is the angle between $\vec{A}$ and $\vec{B}$. (Either angle can be used.)

$$\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$$

$$\vec{A} \bullet (\vec{B} + \vec{C}) \;=\; \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C} \;=\; (\vec{B} + \vec{C}) \bullet \vec{A}$$
$$s(\vec{A} \bullet \vec{B}) \;=\; (s\vec{A}) \bullet \vec{B} \;=\; \vec{A} \bullet (s\vec{B})$$

Magnitude or Length

$$\left| \vec{A} \right| \;=\; \sqrt{\vec{A} \bullet \vec{A}}$$
$$\;=\; \sqrt{A_x^2 + A_y^2 + A_z^2}$$

Cross Product or Outer Product

$$\vec{D} \;=\; \vec{A} \times \vec{B} \text{ is a vector (pseudo-vector). It is perpendicular to both } \vec{A} \text{ and } \vec{B}$$
$$0 \;=\; \vec{D} \bullet \vec{A} \;=\; \vec{D} \bullet \vec{B}$$

$$\left| \vec{D} \right| = \left| A \right| \left| B \right| \sin \theta$$

The direction of $\vec{D}$ is perpendicular to $\vec{A}$ and $\vec{B}$, in the direction of a right hand screw advance when $\vec{A}$ is rotated into $\vec{B}$. (Right hand rule.)

The cross product is not commutative,

$$\vec{A} \times \vec{B} \;=\; -\vec{B} \times \vec{A}$$

$$\vec{D} \;=\; \vec{A} \times \vec{B}$$
$$\;=\; (\, A_y B_z - A_z B_y, \quad A_y B_z - A_z B_y, \quad A_y B_z - A_z B_y \,)$$

Note the cyclic nature of the multiplications in component notation.

Formally, manipulating symbols, we can represent the cross product as a determinate.

$$\vec{D} \;=\; \vec{A} \times \vec{B}$$
$$\;=\; \det \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

Orthogonal Unit (Orthonormal) Vectors
From the properties of the inner product one has:

$$\hat{e}_x \bullet \hat{e}_x \quad = \quad \hat{e}_y \bullet \hat{e}_y \quad = \quad \hat{e}_z \bullet \hat{e}_z \quad = \quad 1$$

$$\hat{e}_x \bullet \hat{e}_y \quad = \quad \hat{e}_x \bullet \hat{e}_z \quad = \quad \hat{e}_y \bullet \hat{e}_z \quad = \quad 0$$

The magnitude of unit vectors cross products are:

$$\left| \hat{e}_x \times \hat{e}_y \right| \quad = \quad \left| \hat{e}_y \times \hat{e}_z \right| \quad = \quad \left| \hat{e}_z \times \hat{e}_x \right| \quad = \quad 1$$

$$\left| \hat{e}_x \times \hat{e}_x \right| \quad = \quad \left| \hat{e}_y \times \hat{e}_y \right| \quad = \quad \left| \hat{e}_z \times \hat{e}_z \right| \quad = \quad 0$$

The cross products are:

$$\hat{e}_x \times \hat{e}_y \quad = \quad 1 \quad = \quad -(\hat{e}_y \times \hat{e}_x)$$

$$\hat{e}_y \times \hat{e}_z \quad = \quad 1 \quad = \quad -(\hat{e}_z \times \hat{e}_y)$$

$$\hat{e}_z \times \hat{e}_x \quad = \quad 1 \quad = \quad -(\hat{e}_x \times \hat{e}_z)$$