Summary of Vector Properties

Basics Quantities - Scalars and Vectors

There are quantities, such as velocity, that have both magnitude and direction. These are vectors. A quantity with only magnitude is a scalar.

Nomenclature

$\vec{A}, \vec{B}, \vec{C}, \vec{D}$	will be vectors
$\lvert\vec{A}\rvert$	is the magnitude or length of the vector $\vec{A}$. It is a scalar.
$\hat{e}_x, \hat{e}_y, \hat{e}_z$	will be a set of orthogonal unit vectors along the x,y,z axes (Some authors use **i,j,k** for these unit vectors.)
A_x, A_y, A_z	are scalars that are the components of the vector $\vec{A}$
s	is a scalar

Component Notation

$$\begin{aligned} \vec{A} &= A_x\hat{e}_x + A_y\hat{e}_y + A_z\hat{e}_z \\ &= (A_x, A_y, A_z) \end{aligned}$$

Scalar Multiplication

$$s\,\vec{A} = s\,(A_x, A_y, A_z) = (sA_x, sA_y, sA_z)$$

$$\begin{aligned} s\,(\vec{A} + \vec{B}) &= s\,\vec{A} + s\,\vec{B} \\ &= (sA_x + sB_x, sA_y + sB_y, sA_z + sB_z) \end{aligned}$$

Addition

$$\begin{aligned} \vec{A} + \vec{B} &= \vec{B} + \vec{A} \\ &= (A_x + B_x, A_y + B_y, A_z + B_z) \end{aligned}$$

Dot Product or Inner Product

$\vec{A} \bullet \vec{B}$ is a scalar.

$$\begin{aligned} \vec{A} \bullet \vec{B} &= \lvert\vec{A}\rvert\,\lvert\vec{B}\rvert \cos\theta \\ &= A_xB_x + A_yB_y + A_zB_z \end{aligned}$$

Where θ is the angle between $\vec{A}$ and $\vec{B}$. (Either angle can be used.)

$$\vec{A} \bullet \vec{B} = \vec{B} \bullet \vec{A}$$

$$\vec{A} \bullet (\vec{B} + \vec{C}) \quad = \quad \vec{A} \bullet \vec{B} + \vec{A} \bullet \vec{C} \quad = \quad (\vec{B} + \vec{C}) \bullet \vec{A}$$

$$s(\vec{A} \bullet \vec{B}) \quad = \quad (s\vec{A}) \bullet \vec{B} \quad = \quad \vec{A} \bullet (s\vec{B})$$

Magnitude or Length

$$\begin{aligned} \left|\vec{A}\right| \quad &= \quad \sqrt{\vec{A} \bullet \vec{A}} \\ &= \quad \sqrt{A_x^2 + A_y^2 + A_z^2} \end{aligned}$$

Cross Product or Outer Product

$\vec{D} \quad = \quad \vec{A} \times \vec{B}$ is a vector (pseudo-vector). It is perpendicular to both $\vec{A}$ and $\vec{B}$

$$0 \quad = \quad \vec{D} \bullet \vec{A} \quad = \quad \vec{D} \bullet \vec{B}$$

$$\left|\vec{D}\right| = |A|\ |B| \sin\theta$$

The direction of $\vec{D}$ is perpendicular to $\vec{A}$ and $\vec{B}$, in the direction of a right hand screw advance when $\vec{A}$ is rotated into $\vec{B}$. (Right hand rule.)

The cross product is not commutative,

$$\vec{A} \times \vec{B} \quad = \quad -\vec{B} \times \vec{A}$$

$$\begin{aligned} \vec{D} \quad &= \quad \vec{A} \times \vec{B} \\ &= \quad (\ A_y B_z - A_z B_y, \quad A_y B_z - A_z B_y, \quad A_y B_z - A_z B_y\ ) \end{aligned}$$

Note the cyclic nature of the multiplications in component notation.

Formally, manipulating symbols, we can represent the cross product as a determinate.

$$\begin{aligned} \vec{D} \quad &= \quad \vec{A} \times \vec{B} \\ &= \quad \det \begin{vmatrix} \hat{e}_x & \hat{e}_y & \hat{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \end{aligned}$$

Orthogonal Unit (Orthonormal) Vectors

From the properties of the inner product one has:

$$\hat{e}_x \bullet \hat{e}_x = \hat{e}_y \bullet \hat{e}_y = \hat{e}_z \bullet \hat{e}_z = 1$$
$$\hat{e}_x \bullet \hat{e}_y = \hat{e}_x \bullet \hat{e}_z = \hat{e}_y \bullet \hat{e}_z = 0$$

The magnitude of unit vectors cross products are:

$$\left|\hat{e}_x \times \hat{e}_y\right| = \left|\hat{e}_y \times \hat{e}_z\right| = \left|\hat{e}_z \times \hat{e}_x\right| = 1$$
$$\left|\hat{e}_x \times \hat{e}_x\right| = \left|\hat{e}_y \times \hat{e}_y\right| = \left|\hat{e}_z \times \hat{e}_z\right| = 0$$

The cross products are:

$$\hat{e}_x \times \hat{e}_y = 1 = -(\hat{e}_y \times \hat{e}_x)$$
$$\hat{e}_y \times \hat{e}_z = 1 = -(\hat{e}_z \times \hat{e}_y)$$
$$\hat{e}_z \times \hat{e}_x = 1 = -(\hat{e}_x \times \hat{e}_z)$$