Basic Physics:
Some Concepts for Geodesy

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I. Introduction

This note addresses some of the basic concepts of physics with emphasis on items needed to understand the fundamentals of geodesy. There are entire books on basic / elementary physics, and these should be consulted for more information. It is assumed that the reader is familiar with vectors. There is a companion note on vectors.

II. The Big Ideas of Physics

There are 3 major ideas that physics uses that are not common in everyday life:

1. Scalars and Vectors

Most quantities are scalars (mass, density) or vectors (position, velocity, force). Vectors are the new, important item. Vectors have a magnitude and direction. It takes one number to specify a scalar. It takes 3 numbers to specify a vector.

2. Conserved Quantities

There are a few things that are conserved - they do not change with time in a closed (isolated) system. Mass, total energy, momentum, and angular momentum are a few of these.

3. Rates of Change

Things are often related to each other through rates of change. This brings in the calculus. Force is related to the rate of change of velocity with respect to time (second derivative of position with respect to time) - that is acceleration. This is Newton's second law. Force is also related to the rate of change of potential energy with respect to position.

There are several other topics that are often covered in an elementary physics text that will not be discussed here. These include waves (water wave, light waves etc.) and the properties of large groups of items - temperature, heat energy etc. Waves are related to oscillations. Oscillations are very common in nature because we are often operating near an equilibrium (balanced forces) condition. Advanced physics covers many other topics such as relativity and quantum mechanics.

III. Newton's Laws of Motion

Isaac Newton published his three laws of motion in 1687. These relate the motion of an object to the forces on it. They do not address the origin of the forces. They just give a prescription for computing motion when the forces are known.

Newton's First Law

In the absence of forces, an object maintains its velocity. If at rest, it remains at rest. If in motion, it remains in motion in the same direction and with the same speed.

Newton's Second Law

The force on a body is proportional to the mass times the acceleration.

$$\vec{F} = m\vec{a}$$

Here the acceleration $\vec{a}$, is the time rate of change of velocity, $\vec{v}$. The velocity is in turn the time rate of change of position, $\vec{x}$. These values have little arrows over them to indicate that they are vectors. A better way to express this equation is

$$\frac{\vec{F}}{m} = \vec{a}$$

which is the way the equation is normally used. The mass is the constant that relates a given force to the rate of change of velocity that is produced.

Newton's Third Law

For every force there is an equal and opposite force. This says that if Item-A acts on Item-B with force $\vec{F}_A$, then Item-B acts on Item-A with a force $\vec{F}_B = -\vec{F}_A$. The total forces in an isolated or closed system are zero. (Sometimes you have to include a lot of things to get all the items that form the closed system.)

IV Conserved Quantities

There are many things that are conserved in physics. The most important one are total energy and momentum. For geodesy and satellite orbits the angular momentum also is important.

1. Total Energy

Energy is a common quantity that we use and discuss in everyday life, but it is also a subtle quantity in physics. It can exist in many forms. The most common form discussed in elementary physics is Kinetic Energy, or energy of motion. It is given by:

$$\begin{aligned} T &= \frac{1}{2} m v^2 \\ &= \frac{1}{2} m \vec{v} \bullet \vec{v} \end{aligned}$$

(There area several different symbols used for kinetic energy, I use T here.) Notice that kinetic energy is a scalar. Energy, in all its many forms, is a scalar.

A second form of energy introduced in elementary physics is potential energy, V. The most common example is the gravitational potential energy. We increase the potential energy of an item if we raise it up. (Move it further from the center of the earth.) Near the earth's surface we can say that raising a mass m from height h_1 to height h_2 increases the gravitational potential energy by:

$$\Delta V = mg(h_2 - h_1) \ .$$

Again energy is a scalar. And also note that only the difference in potential energies is computed. It is very common to have no absolute reference for potential energies. Differences are what are important. The force of gravity on an object is mg. Notice that

$$\vec{F}_g = m\vec{g} = -\frac{\Delta V}{\Delta h} = -\left.\frac{\Delta V}{\Delta h}\right|_{\text{Limit } \Delta h = 0}$$

showing the relationship between the force of gravity and the potential change with respect to position. This concept is valid in general when the second form involving the limit is used. (The spatial gradient operator in more mathematical terms, $\vec{F} = -\vec{\nabla} V$). This makes the object want to move to locations of lower potential energy.

In raising the object, we had to exert a force. The force has a value mg, the 'force of gravity' the earth exerts on the object. We have done work on the object in the technical sense. If we release at height h_2, the object falls. When it reaches the height h_1 it will have lost all the potential energy we added in doing the work raising it. This will show up as kinetic energy. Equating the two, we can find the velocity there:

$$\frac{1}{2}mv^2(h) \quad = \quad mg(h_2 - h).$$

This is just one example of the utility of a conservation law. Notice that I have written the velocity as function of height, v(h). Thus we can use this to find a formula for the velocity at different heights. We still do not know the time history of the object. For that we need to use Newton's second law:

$$\frac{\vec{F}}{m} = \vec{a} = \frac{m\vec{g}}{m} = \vec{g},$$

which tells us that the acceleration is constant, is directed downward, and is equal to the acceleration of gravity. (An important very subtle issue here is in equating the resistance to motion mass - called inertial mass - to the gravitational mass. This is true to a very high degree of accuracy but this wasn't validated by measurement until about 1900.)

** Note Well

In geodesy the gravity potential is not the same as physic's potential energy.

a. It has the opposite sign. This removes the negative sign from the gradient equation.
b. It is potential per unit mass.
c. Rotational effects are included.

There are many other forms of energy. Heat energy is one common place "lost energy" in a system goes. The conservation of energy wasn't proved until the experiments of Joule, Helmholtz and others about heat energy. This was published only in 1847.

2. Momentum or Linear Momentum

The momentum of an object is given as

$$\vec{p} = m\vec{v}.$$

This is a vector. The sum of the momentum of all the objects in a closed system is constant. This is a direct consequence of Newton's third law. Conservation of momentum is commonly used in solving for the dynamics in collisions.

3. Angular Momentum

Angular momentum is related to rotation. It is a vector and commonly denoted as $\vec{L}$. It needs to have a origin of a coordinate system defined to be computed. So it might be named angular momentum with respect to a particular origin, such as the center of the earth. While the values of $\vec{L}$ change with choice of origin, the conservation law holds in all systems. You just have to use a consistent set of values. For rotating systems, the center of rotation is usually used as the origin of the coordinates system.

Angular momentum is defined as using the vector cross product:

$$\vec{L} \quad = \quad m\vec{r}\times\vec{v}$$

where $\vec{r}$ is the vector from the origin to the object, $\vec{v}$ is the velocity and m is the mass. It is a vector perpendicular to the plane formed by $\vec{r}$ and $\vec{v}$ in the direction of a right hand screw rotating r into v. If $\vec{r}$ is parallel $\vec{v}$ then the angular momentum is zero. The magnitude of the angular momentum is maximum when $\vec{r}$ is perpendicular to $\vec{v}$, as it is in circular motion.

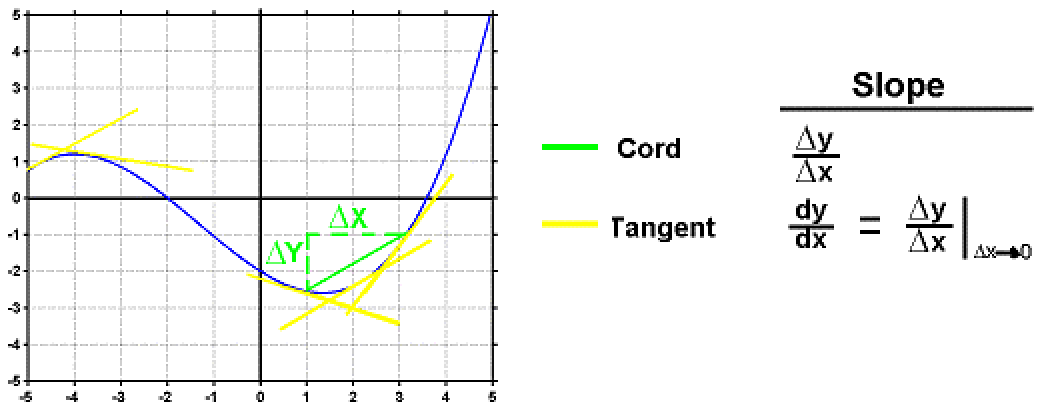
V. Rates of Change (Derivatives) and Integrals

There are many places in physics where the rates of change occur. This is the derivative of calculus. It is not hard to understand. Measure the speed of a car by placing two sensor cables across the road as was done before the invention of the radar gun. The time difference in the pulses as the car crosses the cables is the separation, s, divided by the velocity averaged over the separation of the two sensor cables,

$$\Delta t = \frac{s}{v_{avg}},$$

so the average velocity is $s/\Delta t$. This is an average velocity. As the cables come closer, the measurement becomes closer to the instantaneous velocity of the car. This limit is the derivative.

Plot the graph of a function as shown below. The cord, the line between two points, has a slope that is the like the average velocity. As the length between the contact points of the cord get shorter, the cord becomes the tangent line to the curve. There will be a separate tangent line at each point. The slope of the tangent line is the derivative of the curve. This exists at each point.



Derivative of a Curve

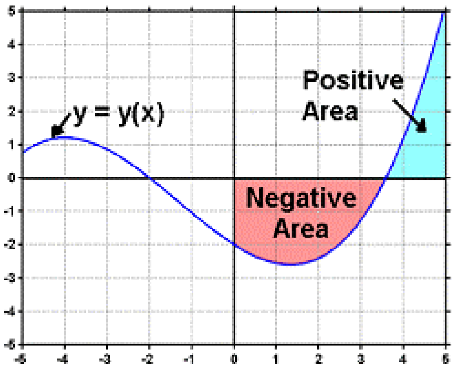
Often we need to do the inverse of taking a derivative. This is done with the integral of the curve. The fundamental theory of calculus says that "the integral is the inverse operation of the derivative".

$$f(x) = \int \frac{df}{dx}\, dx$$

The integral is the area "under" the curve. In mathematical language, the integral is the limit of the areas of a series of boxes of width Δx and with height being f(x) at the midpoint.

$$S = \sum_i f(x_i)\, \Delta x$$

The integral is the limit of this sum where the width of the boxes goes to zero and the number of boxes goes to infinity. The area under the curve is thus the signed sum of the area, where the area below the axis is negative and that above is positive.



$$Y = \int_0^5 y(x)\,dx$$

$$= \text{Area "Under" Curve}$$

Integral — Area Under Curve

There are two different types of integrals. The integral above is called a definite integral. It is between two specific x values. You can also take the integral using a variable for the upper limit. In this case you get a function, not a number.

The further details of the calculus can be found in many textbooks. Derivatives are usually easy to compute. There are a series of runs that can be applied. The same is true of integral, but they can be harder to find in a table. You sometimes have to manipulate them to get them into a standard form.

VI The Gradient

In physics, and in particular in considering the gravity field of the earth, a multi-dimensional version of the derivative is used. This is called the gradient operator and is usually written as $\vec{\nabla}$. It acts like a vector. A brief overview of this operator is given here along with the few important properties for the geodesy application. For more details, look at mathematics, physics or engineering text books.

Mathematically the gradient can be described in several ways. One of the easiest is in terms of Cartesian components. (See the technical note on vectors for details on the Cartesian representation of a vector.)

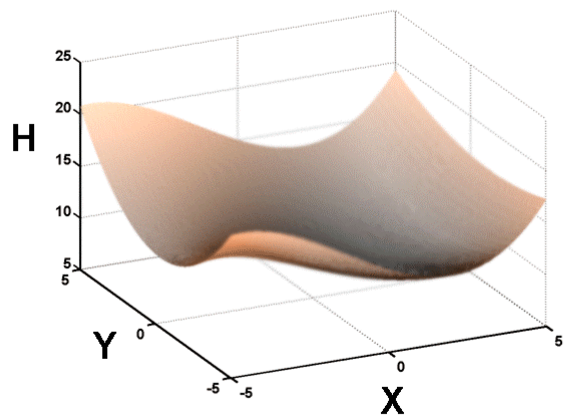
If a function depends on two or more variables, such as the position values of latitude and longitude, then you can take the derivative with respect to each of these independent variables assuming that the other variable is constant. This is called the partial derivative. If f = f(x, y) then the partial derivative with respect to x is written as

$$\frac{\partial f(x,y)}{\partial x} = \frac{\partial f}{\partial x},$$

and the partial with respect to the second independent variable, y, as

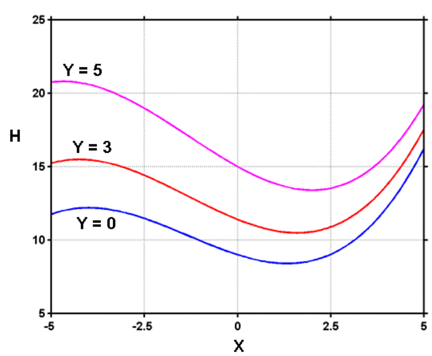
$$\frac{\partial f}{\partial y}.$$

Consider the following diagram of the height in some gentle rolling prairie.

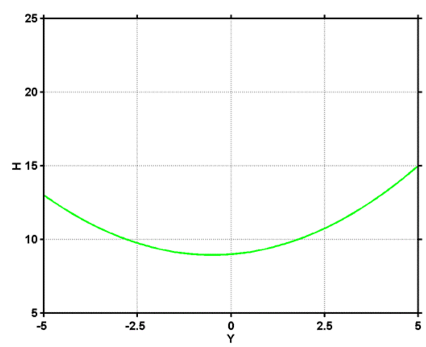


Height In Two Dimensions

The height is our function f. It depends on two horizontal values called X and Y here. For these fixed values we can plot graphs in 1 dimension.

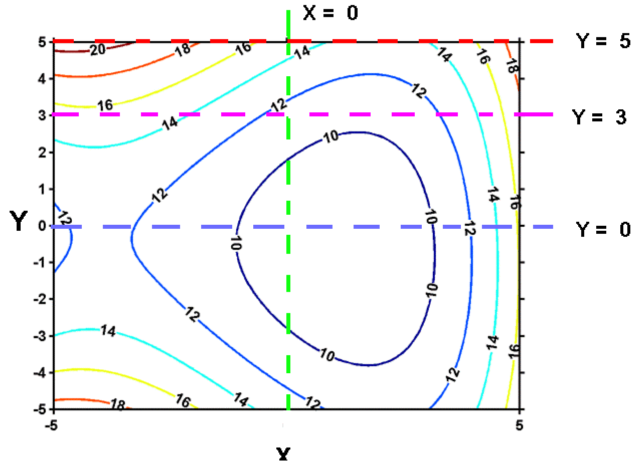


Height Cuts Through Constant Y Planes



Height Cut Through Constant X Plane

Below is a contour plot of the height.   This shows lines of constant height or level. They are also levels of equal gravitational potential energy. The cuts for the four lines plotted are marked with dotted lines.



Height Contour Plot
Showing Lines of Cut Plots

The derivatives (slope of the tangent lines) of the height along the cuts are partial derivatives of the two dimensional figure. There is a derivative at each value of X in the left figure. Notice that the Y=0 cut is only a function of X. Now we could have just as easily taken the Y cut at a value of Y=3 or Y=5, which are the also shown. . Those curves are different, the derivative (partial derivative of the 2-dimensional function) is different from the Y=0 cut. So a partial derivative is labeled with the values of both the independent variables. The partials are functions over the same domain as the original function. And there are two of them in the two dimensional case. For the three dimension case there are three partial derivatives.

These two different partials are combined to form a vector, the gradient. The gradient of f, is given by:

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{e}_x + \frac{\partial f}{\partial y}\hat{e}_y$$
$$= \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right)$$

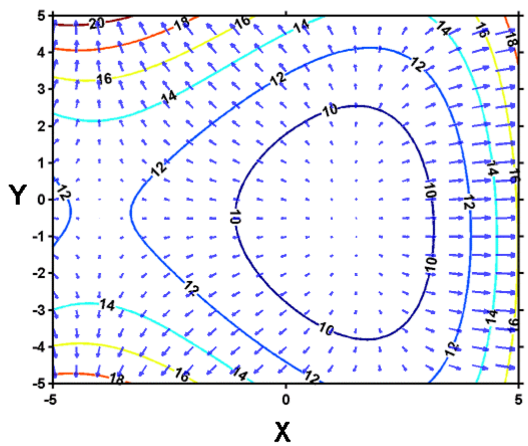
where $\hat{e}_x$ is the unit vector in the x direction and $\hat{e}_y$ is the unit vector in the y direction. The second line is just another notation for the same vector. We see that the gradient operator takes a scalar function and produces a vector.

The meaning of the gradient is simple: it is a vector giving the slope of the two dimensional (or three dimensional) surface. Here in two dimensions it is fairly easy to interpret. It points in the direction of the maximum increase of the function. The length of the vector is equal to the slope in that direction.

Now below is the same contour plot, with small vectors added for the gradient. At each point there is a gradient. We see several things.

1. The magnitude of the gradient is related to the slope of the surface. Where the slope is large, the contour lines are close together. There the gradient is largest.

2. Note that the gradients are always perpendicular to the contour lines. This is a key feature for geodesy.



Height Contour Plot
Showing Gradient Vectors

The gradient point uphill in the direction of maximum slope. This is a gradient of height, but that is also a gradient of gravitational potential energy. Because the force is given by

$$\vec{F} = -\vec{\nabla} V$$

with a negative sign, the force is downhill. Where the gradient is high, it takes more energy to climb the hill. Or water will flow downhill faster. Where the gradient is zero there is no force in the X-Y plane.