5B.1 A Fully Conserved Adjustment Scheme for Stabilization of Hydrographic Profiles

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Abstract

Hydrographic data, be it observational or averaged data, contain substantial regions having vertical density inversions. A new analytical conserved adjustment scheme has been developed on the base of conservation of heat, salt, and static stability for the whole water column with a predetermined \((T, S)\) adjustment ratio. A set of well-posed combined linear and nonlinear algebraic equations has been established and is solved using the Newton’s method. This new scheme can be used for ocean hydrographic data analysis and data assimilation.

1. Introduction

Raw and averaged observational hydrographic data contain substantial regions having vertical density inversions. For example, Jackett and McDougall (1995) found that the annually averaged field of the Ocean Atlas of Levitus (1982) had more than 44% of the casts possessing static instability at least at one level. Here, the word ‘cast’ is used to denote a pair of vertical temperature and salinity profiles. A widely used concept for static stability \((E)\) is defined by Lynn and Reid (1968) as “the individual density gradient by vertical displacement of a water parcel (as opposed to the geometric density gradient).” For discrete samples \((T_k, S_k)\) at depth \(z_k, k = 1, 2, \ldots, K\) (k increasing downward), the density difference between two adjacent levels is taken after one is adiabatically displaced to the depth of the other. Computationally, \(E_k\) is calculated by

\[
E_k = \rho(S_{k+1}, T_{k+1}, z_{k+1}) - \rho(S_{k}, T_{k}, z_{k}), \quad k = 1, 2, \ldots, K-1
\]  

where \(\rho(S_{k+1}, T_{k+1}, z_{k+1})\) is the local potential density of the lower of the two adjacent levels between \(z_k\) and \(z_{k+1}\) with respect to the upper of the two adjacent levels \(z_k\); and \(\rho\) is the in-situ density to the depth of the upper of the two adjacent levels \(z_k\). The density inversion is defined by the occurrence of negative value of \(E_k\). The minimum static stability is represented by \(E_k = 0\). It is not always possible to reach zero exactly due to the precision limitations of the temperature and salinity values used. As a result, the minimum value for the static stability is given by

\[
E_k \geq E_{\text{min}}, \quad k = 1, 2, \ldots, K
\]  

where \(E_{\text{min}}\) is the reference value for the minimum static stability. If static instability occurs in an observed or averaged hydrographic cast [i.e., (2) does not satisfy], this profile need to be adjusted.

The National Ocean Data Center (NODC) uses a local interactive \((T, S)\) separated adjustment method, which is based on the method proposed by Jackett and McDougall (1995) (hereafter referred to JM method) to minimally alter climatological temperature and salinity profiles to achieve a stable water column everywhere in the world ocean. Before deciding which level to change, the values of \(\partial T / \partial z\) and \(\partial S / \partial z\), the gradient of temperature and salinity between two adjacent levels involve in the instability, are examined. This helps determine if the temperature or salinity profile, or both, are to be changed to stabilize the density field. If \(\partial T / \partial z < 0, \partial S / \partial z < 0\), only temperature is changed; If \(\partial T / \partial z > 0, \partial S / \partial z > 0\), only salinity is changed; If \(\partial T / \partial z < 0, \partial S / \partial z > 0\), both temperature and salinity fields are adjusted with a local linear trend test (Locarnini et al. 2006). Here, z-axis points upward. The principal is to stabilize the hydrographic profiles with minimum adjustment.
The benefit of using the JM method can be easily identified from comparison between two ocean atlases: Ocean Atlas of Levitus (1982) (without the JM adjustment) and World Ocean Atlas 2005 (Locarnini et al. 2006) (with the JM adjustment). Both atlases consist of annually and monthly averaged vertical profiles of temperature and salinity on a global 1° \( \times 1° \) grid at 33 vertical levels. The Ocean Atlas of Levitus (1982) has considerable casts possessing static instability. However, the World Ocean Atlas 2005 contains no profile possessing static instability.

Although eliminating the static instability, the JM method does not require the conservation of heat and salt. Since one of ocean’s important roles in the earth’s climate is heat transport, the adjustment without taking into account of heat conservation may lead to error in estimating the ocean’s impact on global climate change. In this study, a new conserved scheme is developed to simultaneously adjust the temperature and salinity profiles from \( (T_k, S_k) \) to \( (T_k + \Delta T_k, S_k + \Delta S_k) \). A set of \( 2K \) algebraic (linear and nonlinear) equations are established to get \( (\Delta T_k, \Delta S_k) \) on the base of heat and salt conservation, predetermined \( (\Delta T_k / \Delta S_k) \) ratios (or called adjustment ratios) for all levels, and removal of static instability by adjusting \( E_k \) to \( E_k + \Delta E_k \) with a combined conservation and non-uniform increment treatment.

2. Unconserved Adjustment

An example as described in Appendix B of Locarnini et al. (2006) is used for illustration. The area chosen for this example is the one-degree latitude-longitude box centered at 53.5°S -171.5°E from a previous version of the World Ocean Atlas 1998 (WOA98). This is on the New Zealand Plateau, with a bottom depth below 1000 m and above 1100 m. The month is October, during the early austral summer. There is no temperature or salinity data within the chosen one-degree box. Thus the objectively analyzed values in this one-degree box will be dependent on the seasonal objectively analyzed field and the data in nearby one-degree grid boxes. There is much more temperature data than salinity data on the New Zealand plateau for October. This contributes to six small (on the order of \( 10^{-2} \) kg m\(^{-3}\)) inversions in the local potential density field calculated from objectively analyzed temperature and salinity fields (Table 1). After using the JM method, the original and adjusted profiles \( \{T_k, S_k, k = 1, 2, \ldots, K\} \) are as shown in Fig. 1, and the adjusted temperature and salinity profiles are listed in Table 2. Readers are referred to Appendix B of Locarnini et al. (2006) for detailed information on the stabilization procedures. The relative root mean adjustment (RRMA) can be represented by

\[
RRMA = \frac{\sqrt{\frac{1}{K} \sum_{k=1}^{K} (\Delta T_k)^2}}{\max(T_k) - \min(T_k)} + \frac{\sqrt{\frac{1}{K} \sum_{k=1}^{K} (\Delta S_k)^2}}{\max(S_k) - \min(S_k)}
\]

\[= 0.0712. \tag{3}\]

The total heat and salt changes of the water column within this \( 1° \times 1° \) grid box are estimated by

\[
\Delta Q = A \rho_c c_p \int_{-H}^{0} \Delta T dz, \quad \Delta(Salt) = A \int_{-H}^{0} \Delta S dz,
\]

where \( \rho_c (=1028 \text{ kg m}^{-3}) \) is the characteristic density, \( c_p (= 4002 \text{ J kg}^{-1} \text{ K}^{-1}) \) is the specific heat for the sea water, \( H = 1000 \text{ m} \), and \( A \) is the area of the grid box,

\[A = \left( \frac{\pi}{180} R \right)^2 \cos \varphi,\]

where \( R (= 6370 \text{ km}) \) is the earth radius, and \( \varphi (= 53.5°) \) is the latitude of the grid box. The temperature and salinity adjustments \( (\Delta T, \Delta S) \) are obtained by comparison between Table 1 and Table 2, the heat and salt changes of the water column for this grid box are calculated by

\[
\Delta Q = -7.0411 \times 10^3 J, \quad \Delta \text{(salt)} = -0.5443 \times 10^4 \text{kg}.
\]

Since one of the ocean’s important roles in the earth’s climate is transporting heat from low to high latitudes, nontrivial heat and salt losses show that the unconserved adjustment may change heat transport and in turn affect the overturning thermohaline circulation.

3. Stabilization

The stabilization process is divided into three parts: (a) stability increasing at unstable levels, (b) stability decreasing at stable levels, (c) normalization for conservation of stability for the cast. Let static instability occur at level \( k_1, k_2, \ldots, k \) [i.e., satisfies the inequality (2)], the static stability \( E_{k_i} \) is increased to its marginal stability value \( (E_{k_i}^*) \),

\[
E_{k_i}^* = E_{max}, \tag{4}
\]

with increments of \( \Delta E_{k_i} = E_{max} - E_{k_i} \).

Such an increase of stability will be compensated by the decrease of stability at neighboring levels \( k \pm m (m = 1, 2, \ldots) \) with skipping the unstable levels until reaching the top and bottom of the profile,
The static stabilities for the whole profile before and after the adjustment is calculated by
\[ E_{k,n}^- = \begin{cases} E_{k,n} - \Delta E_{k,i} / 2^{n+1} & \text{if } E_{k,n} - \Delta E_{k,i} / 2^{n+1} \geq E_{\min} \\ E_{\min} & \text{if } E_{k,n} - \Delta E_{k,i} / 2^{n+1} < E_{\min} \end{cases} \]  
(5)

5. Newton Method

Let the temperature and salinity adjustment be represented by a 2K-dimensional vector \( P \),
\[
\begin{bmatrix}
    p_1 \\
    p_2 \\
    p_3 \\
    \vdots \\
    p_M
\end{bmatrix} =
\begin{bmatrix}
    \Delta T_1 \\
    \Delta S_1 \\
    \Delta T_2 \\
    \vdots \\
    \Delta S_K
\end{bmatrix}, \quad M = 2K.
\]  
(13)

The algebraic equations (10), (11), (12), (8) [note that we put (8) at the last] can be represented by
\[
F(P) = 0,
\]  
(14)

where \( F \) has the dimension of 2K. The classical Newton Method for approximating a desired solution \( P \) to (14) is formally defined by the iteration
\[
\left( \begin{array}{c}
    p_1 \\
    p_2 \\
    \vdots \\
    p_M
\end{array} \right)_{j+1} = \left( \begin{array}{c}
    \Delta T_1 \\
    \Delta S_1 \\
    \Delta T_2 \\
    \vdots \\
    \Delta S_K
\end{array} \right)_j.
\]  
(15)

4. Constraints for Temperature and Salinity Adjustment

Two types of constraints are used: conservation and predetermined adjustment ratios. Conservation of heat and salt for the adjustment can be represented by
\[
\int_0^0 \Delta T dz = 0, \quad \int_{-K}^0 \Delta S dz = 0,
\]  
(9)

which can be discretized by
\[
\sum_{i=1}^{K-1} \left( \frac{\Delta T_i + \Delta T_{i+1}}{2} \right) (z_k - z_{k+1}) = 0, \quad (10)
\]
\[
\sum_{i=1}^{K-1} \left( \frac{\Delta S_i + \Delta S_{i+1}}{2} \right) (z_k - z_{k+1}) = 0. \quad (11)
\]

The adjustment ratios ( \( \Delta T_i / \Delta S_i = \gamma_i \), the negative sign is used due to opposite effects of T, S on the density, \( \gamma_i > 0 \) ) are predetermined empirically for N-1 levels.
\[
\Delta T_i + \gamma_i \Delta S_i = 0, \quad k = 1, 2, ..., K-1. \quad (12)
\]

Eqs.(10), (11), (12), and (8) represent a set of 2K algebraic equations for ( \( \Delta T_i, \Delta S_i \) ), \( k = 1, 2, ..., K \). Among them, (8) is nonlinear and (10), (11), (12) are linear. The Newton method is used to solve the set of 2K algebraic equations.
The Jacobian matrix (18) has the following format with many zero elements,
\[
J_f(P^{(j)}) = \begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1M} \\
  a_{21} & a_{22} & \cdots & a_{2M} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{M1} & a_{M2} & \cdots & a_{MM}
\end{bmatrix},
\]
where the \( M \times M \) elements are given in (A1) of Appendix A. The Jacobian matrix (18) has the following format with many zero elements,
\[
\begin{bmatrix}
  * & 0 & * & 0 & \cdots & * & 0 \\
  0 & * & 0 & 0 & \cdots & 0 & * \\
  * & 0 & 0 & 0 & \cdots & 0 & 0 \\
  * & * & * & 0 & \cdots & 0 & 0 \\
  0 & 0 & * & 0 & \cdots & 0 & 0 \\
  0 & 0 & * & * & \cdots & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
  0 & 0 & 0 & \cdots & * & * & 0 \\
  0 & 0 & 0 & \cdots & * & * & *
\end{bmatrix},
\]
where nonzero elements are indicated by the symbol ‘*’. The vector \( b \) in the righthand-side of (16) has the following components:
\[
b_1 = 0, \quad b_2 = 0, \quad b_3 = 0, \\
b_4 = E_{2}^0 - \rho \left(S_k + \Delta S_k^{(1)}, T_k + \Delta T_k^{(1)}, z_k\right) \\
+ \rho \left(S_k + \Delta S_k^{(1)}, T_k + \Delta T_k^{(1)}, z_k\right), \\
b_5 = 0, \\
b_6 = E_{2}^0 - \rho \left(S_k + \Delta S_k^{(1)}, T_k + \Delta T_k^{(1)}, z_k\right) \\
+ \rho \left(S_k + \Delta S_k^{(1)}, T_k + \Delta T_k^{(1)}, z_k\right), \\
b_{M-1} = 0, \\
b_M = E_{k-1}^0 - \rho \left(S_{k-1} + \Delta S_{k-1}^{(1)}, T_{k-1} + \Delta T_{k-1}^{(1)}, z_{k-1}\right) \\
+ \rho \left(S_{k-1} + \Delta S_{k-1}^{(1)}, T_{k-1} + \Delta T_{k-1}^{(1)}, z_{k-1}\right).
\]

Substitution of (26) into (1) gives static stability after first iteration \( E_k^{(1)} \). If
\[
E_k^{(1)} \geq E_{min}, \quad k = 1, 2, ..., K,
\]
the adjustment stops. Otherwise, the iteration continues, i.e., the linear algebraic equation (16) is solved after using \( P^{(1)} \) from (26). Addition of the solution \( \Delta T_k^{(1)} \) to \( T_k^{(1)} \) leads to \( T_k^{(2)} \). If there is no static instability, the adjustment stops. Otherwise, the iteration continues until the static instability is eliminated. For the hydrographic cast listed in Table 1, three iterations are needed to eliminate the static instability. Tables 3 and 4 list the values of \( \{S_k, T_k\} \) at the each iteration. They show the high efficiency of this method for elimination of static instability in hydrographic cast. Fig. 2 shows the original and adjusted profiles \( \{S_k, T_k\} \) at each iteration. They show the high efficiency of this method for elimination of static instability in hydrographic cast. Fig. 2 shows the original and adjusted profiles \( \{S_k, T_k\} \), \( k = 1, 2, ..., K \). The heat and salt are conserved for the whole water column with the relative root-mean adjustment
\[
RRMA = 0.0482.
\]
Comparing (29) to (3), we may find that this analytical conserved adjustment scheme has a smaller RRMA (0.0482) than the JM method (0.0712).

## 7. Application to Data Assimilation in Ocean Modeling

Data assimilation is required in operational ocean data access and retrieval (Sun 1999). It is to blend the
modeled variable \((x_m)\) with observational data \((y_o)\) (e.g., Lozano et al. 1996, Chu et al. 2004),

\[
x_o = x_n + W \cdot [y_o - H(x_o)],
\]

(30)

where \(x_o\) is the assimilated variable; \(H\) is an operator that provides the model’s theoretical estimate of what is observed at the observational points, and \(W\) is the weight matrix. Difference among various data assimilation schemes such as optimal interpolation (e.g., Barron and Kara 2006), Kalman filter (e.g., Galanis et al. 2006), and variation methods (e.g., Tang and Kleeman 2004) is the different ways to determine the weight matrix \(W\). The data assimilation process (30) can be considered as the average (in a generalized sense) of \(x_m\) and \(y_o\). In ocean \((T, S)\) data assimilation, the observational data \((y_o)\) may contain several casts, which are statically stable. The model profile \((x_m)\) is also statically stable since convective adjustment (Bryan 1969) is usually conducted at each time step.

False static stability may be generated after \((T, S)\) data assimilation [i.e., performing (30)]. For example, 10-day JPL Estimating the Circulation and Climate of the Ocean (ECCO) \((T, S)\) fields centered on 31 December 2008 (download from the website: http://ecco.jpl.nasa.gov/external/) show considerable portion (11.57%) of profiles are statically unstable (Fig. 3). Here, the National Ocean Data Center’s criterion for flagging out statically unstable profiles,

\[
E(k) < \begin{cases} 
-0.03 & (0 \geq z_i \geq -30 \text{ m}) \\
-0.02 & (-30 \text{ m} > z_i \geq -400 \text{ m}) \\
0 & (-400 \text{ m} > z_i) 
\end{cases},
\]

(31)

is used. Since such a false static instability is due to the blending of observational data with the model data, not a real instability. Use of the convective adjustment scheme may over-correct the profiles.

To illustrate this, we discuss the existing convective adjustment schemes in ocean models. The various convective adjustment schemes are based on the same original idea (e.g., Bryan 1969): whenever a water column is statically unstable, temperature and salinity are vertically adjusted to make the water column neutrally stable, with heat and salt conserved in the process. The adjustment takes an iterative approach. The iteration continues between all adjacent levels until the static instability is removed in the whole water column. Because the adjustment acts on only neighboring points, the number of iterations required to reach the final stable state is infinite for a given unstable profile (Smith 1989). In practice, however, the number of iteration is always finite, and this leads to some residual instabilities (Killworth 1989).

Several algorithms were developed to remove these residual static instabilities such as the implicit vertical diffusion scheme (Cox 1984; Killworth 1989) and the complete adjustment scheme (Yin and Sarachik 1994). The former tests the static stability between the vertically adjacent levels and, if unstable, the vertical diffusivity is set to a large value (convective diffusivity) in order to smooth out the instability. The latter is to determine the upper and lower boundaries of each adjusted region while keeping the instantaneous adjustment within each unstable region. Yin and Sarachik (1994) showed that the complete convective adjustment scheme is more efficient than the implicit vertical diffusion scheme and guarantees a complete removal of static instability of a water column at each time step. For the same example as described in Section 2, the complete convective adjustment scheme removes the static instabilities (Fig. 4) with the relative root-mean adjustment,

\[
\text{RRMA} = 0.2192.
\]

This value is 4.5 times larger than that (0.0482) using the analytical adjustment method.

8. Conclusions

A new analytical conserved adjustment scheme is developed to eliminate static instability of raw and averaged observational hydrographic data. This method adjusts the temperature and salinity profiles \{\(\Delta T_k, \Delta S_k\), \(k = 1, 2, ..., K\) \} simultaneously and efficiently on the base of three types of constraints: (a) heat and salt conservation, (b) predetermined \((\Delta T_k / \Delta S_k)\) ratios (or called adjustment ratios) for all levels, and (c) removal of static instability by adjusting the static stability with a combined conservation and non-uniform increment treatment. With these constraints, a set of \(2K\) combined linear/nonlinear algebraic equations are established for \{\(\Delta T_k, \Delta S_k\) \}. Among them, \((K + 1)\) algebraic equations are linear, \((K - 1)\) equations are nonlinear. The Newton’s method is used to solve this set of equations with few steps of iteration. This scheme has very small relative root-mean square adjustment compared to the existing methods.

This scheme has three features: (1) conservation of heat and salt, (2) removal of static instabilities with small \((T, S)\) adjustments, and (3) analytical form. With these features, it is can be widely used in ocean \((T, S)\) data analysis. Besides, ocean data assimilation may cause false static instabilities. Since this instability is not real, minimal adjustment to stabilize the cast is ideal. The analytical
conserved adjustment scheme can be used in ocean (T, S) data assimilation.

Acknowledgments

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Appendix A Elements of Jacobian Matrix (19)

The elements of $M \times M$ Jacobian matrix (19) are given by

$$a_{11} = \frac{\Delta z_1}{2}, a_{12} = \frac{\Delta z_2}{2}, ..., a_{1,M-1} = \frac{\Delta z_{M-1}}{2},$$

$$a_{1,M-1} = \frac{\Delta z_{M-1}}{2}, a_{12} = a_{14} = a_{16} = ... = a_{1M} = 0,$$

$$a_{22} = \frac{\Delta z_1}{2}, a_{34} = \frac{\Delta z_2}{2}, ..., a_{2,M-1} = \frac{\Delta z_{M-1}}{2},$$

$$a_{2,M-1} = \frac{\Delta z_{M-1}}{2}, a_{21} = a_{23} = a_{25} = ... = a_{2M-1} = 0,$$

$$a_{31} = 1, a_{32} = \gamma_1, a_{33} = a_{34} = a_{35} = ... = a_{3M} = 0$$

$$a_{41} = \frac{\partial \rho}{\partial T} \left( S_1 + \Delta S^{(j)}_1, T_1 + \Delta T^{(j)}_1, z_1 \right)$$

$$= -\rho \left( S_1 + \Delta S^{(j)}_1, T_1 + \Delta T^{(j)}_1, z_1 \right) \alpha \left( S_1 + \Delta S^{(j)}_1, T_1 + \Delta T^{(j)}_1, z_1 \right),$$

$$a_{42} = \frac{\partial \rho}{\partial S} \left( S_1 + \Delta S^{(j)}_1, T_1 + \Delta T^{(j)}_1, z_1 \right)$$

$$= \rho \left( S_1 + \Delta S^{(j)}_1, T_1 + \Delta T^{(j)}_1, z_1 \right) \beta \left( S_1 + \Delta S^{(j)}_1, T_1 + \Delta T^{(j)}_1, z_1 \right),$$

$$a_{43} = -\frac{\partial \rho}{\partial T} \left( S_1 + \Delta S^{(j)}_1, T_1 + \Delta T^{(j)}_1, z_1 \right)$$

$$= -\rho \left( S_1 + \Delta S^{(j)}_1, T_1 + \Delta T^{(j)}_1, z_1 \right) \alpha \left( S_1 + \Delta S^{(j)}_1, T_1 + \Delta T^{(j)}_1, z_1 \right),$$

$$a_{44} = \frac{\partial \rho}{\partial S} \left( S_1 + \Delta S^{(j)}_1, T_1 + \Delta T^{(j)}_1, z_1 \right)$$

$$= \rho \left( S_1 + \Delta S^{(j)}_1, T_1 + \Delta T^{(j)}_1, z_1 \right) \beta \left( S_1 + \Delta S^{(j)}_1, T_1 + \Delta T^{(j)}_1, z_1 \right),$$

$$a_{45} = a_{46} = ... = a_{4M} = 0,$$

$$a_{53} = 1, a_{54} = \gamma_2,$$

$$a_{55} = a_{52} = a_{53} = a_{56} = a_{57} = ... = a_{5M} = 0,$$

$$a_{63} = \frac{\partial \rho}{\partial T} \left( S_1 + \Delta S^{(j)}_1, T_2 + \Delta T^{(j)}_1, z_2 \right)$$

$$= -\rho \left( S_1 + \Delta S^{(j)}_1, T_2 + \Delta T^{(j)}_1, z_2 \right) \alpha \left( S_1 + \Delta S^{(j)}_1, T_2 + \Delta T^{(j)}_1, z_2 \right),$$

$$a_{64} = \frac{\partial \rho}{\partial S} \left( S_1 + \Delta S^{(j)}_1, T_2 + \Delta T^{(j)}_1, z_2 \right)$$

$$= \rho \left( S_1 + \Delta S^{(j)}_1, T_2 + \Delta T^{(j)}_1, z_2 \right) \beta \left( S_1 + \Delta S^{(j)}_1, T_2 + \Delta T^{(j)}_1, z_2 \right),$$

$$a_{65} = -\frac{\partial \rho}{\partial T} \left( S_1 + \Delta S^{(j)}_1, T_1 + \Delta T^{(j)}_1, z_1 \right)$$

$$= -\rho \left( S_1 + \Delta S^{(j)}_1, T_1 + \Delta T^{(j)}_1, z_1 \right) \alpha \left( S_1 + \Delta S^{(j)}_1, T_1 + \Delta T^{(j)}_1, z_1 \right),$$

$$a_{66} = a_{62} = a_{67} = a_{68} = ... = a_{6M} = 0,$$

$$a_{M-1,1} = 1, a_{M-1,M-2} = \gamma_{M-1},$$

$$a_{M-1,2} = a_{M-1,4} = ... = a_{M-1,M-4} = a_{M-1,M-1} = a_{M-1,M} = 0,$$

$$a_{M,M-1} = \frac{\partial \rho}{\partial T} \left( S_{k-1} + \Delta S^{(j)}_{k-1}, T_{k-1} + \Delta T^{(j)}_{k-1}, z_{k-1} \right)$$

$$= -\rho \left( S_{k-1} + \Delta S^{(j)}_{k-1}, T_{k-1} + \Delta T^{(j)}_{k-1}, z_{k-1} \right) \alpha \left( S_{k-1} + \Delta S^{(j)}_{k-1}, T_{k-1} + \Delta T^{(j)}_{k-1}, z_{k-1} \right),$$

$$a_{M,M-2} = \frac{\partial \rho}{\partial S} \left( S_{k-1} + \Delta S^{(j)}_{k-1}, T_{k-1} + \Delta T^{(j)}_{k-1}, z_{k-1} \right)$$

$$= \rho \left( S_{k-1} + \Delta S^{(j)}_{k-1}, T_{k-1} + \Delta T^{(j)}_{k-1}, z_{k-1} \right) \beta \left( S_{k-1} + \Delta S^{(j)}_{k-1}, T_{k-1} + \Delta T^{(j)}_{k-1}, z_{k-1} \right),$$

$$a_{M-1,M} = -\frac{\partial \rho}{\partial T} \left( S_{k} + \Delta S^{(j)}_{k}, T_{k} + \Delta T^{(j)}_{k}, z_{k} \right)$$

$$= -\rho \left( S_{k} + \Delta S^{(j)}_{k}, T_{k} + \Delta T^{(j)}_{k}, z_{k} \right) \alpha \left( S_{k} + \Delta S^{(j)}_{k}, T_{k} + \Delta T^{(j)}_{k}, z_{k} \right),$$

$$a_{k,M} = a_{k2} = a_{k3} = a_{k6} = a_{k7} = ... = a_{kM} = 0.$$
\[ a_{sm} = - \frac{\partial \rho (S_k + \Delta S_k^{(i)}, T_k + \Delta T_k^{(i)}, z_k)}{\partial S} \]
\[ = - \rho (S_k + \Delta S_k^{(i)}, T_k + \Delta T_k^{(i)}, z_k) \times \beta (S_k + \Delta S_k^{(i)}, T_k + \Delta T_k^{(i)}, z_k), \]
\[ a_{1,m} = a_{2,m} = \ldots = a_{m-A,m} = 0. \quad (A1) \]

References


Table 1. Grid-box 171.5°E, 53.5°S WOA98 profiles before stabilization (from Locarnini et al. 2006, Table B1). Here the symbol ‘*’ in the last column indicates the static instability.

<table>
<thead>
<tr>
<th>k</th>
<th>Depth (m)</th>
<th>T (°C)</th>
<th>S (ppt)</th>
<th>$\rho(S_{x,z},T_{x,z})$ (kg m$^{-3}$)</th>
<th>$\rho(S_{x},T_{x},z_{x})$ (kg m$^{-3}$)</th>
<th>$E_x$ (kg m$^{-3}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>7.1667</td>
<td>34.4243</td>
<td>26.9476</td>
<td>26.9423</td>
<td>0.0054</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
<td>7.1489</td>
<td>34.4278</td>
<td>26.8982</td>
<td>26.9939</td>
<td>-0.0957*</td>
</tr>
<tr>
<td>3</td>
<td>20</td>
<td>7.0465</td>
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Table 2. Grid-box 171.5°E, 53.5°S WOA98 profiles after stabilization using the JM method (from Locarnini et al. 2006, Table B2).

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Table 3. Change of \( (T_k + \Delta T_k^{(j)}) \) (°C) at each iteration using the Newton’s method. It is noted that the iteration converges at the third iteration.

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Table 4. Change of \((S_k + \Delta S_k^{(j)})\) (ppt) at each iteration using the Newton’s method. It is noted that the iteration converges at the third iteration.

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Fig. 1. Original (dashed) and adjusted (solid) profiles temperature ($T_k$), salinity ($S_k$), and static stability ($E_k$) at the grid box 171.5°E, 53.5°E using the JM method (Locarnini et al. 2006).
Fig. 2. Original (dashed) and adjusted (solid) profiles temperature ($T_k$), salinity ($S_k$), and static stability ($E_k$) at the grid box 171.5°E, 53.5°E using the analytical conserved method proposed in this paper.
Fig. 3. Distribution of statically unstable casts in the JPL-ECCO 10-day data centered on December 31, 2008. The data were produced by a data assimilation system.
Fig. 4. Original (dashed) and adjusted (solid) profiles temperature ($T_k$), salinity ($S_k$), and static stability ($E_k$) at the grid box 171.5°E, 53.5°E using the complete convective adjustment method (Yin and Sarachik Locarnini 1994).