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Map Projections A Working Manual

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MAP PROJECTIONS

A WORKING MANUAL

By JOHN P. SNYDER

ABSTRACT

After decades of using only one map projection, the Polyconic, for its mapping program, the U.S. Geological Survey (USGS) now uses several of the more common projections for its published maps. For larger scale maps, including topographic quadrangles and the State Base Map Series, conformal projections such as the Transverse Mercator and the Lambert Conformal Conic are used. Equal-area and equidistant projections appear in the *National Atlas.* Other projections, such as the Miller Cylindrical and the Van der Grinten, are chosen occasionally for convenience, sometimes making use of existing base maps prepared by others. Some projections treat the Earth only as a sphere, others as either ellipsoid or sphere.

The USGS has also conceived and designed several new projections, including the Space Oblique Mercator, the first map projection designed to permit mapping of the Earth continuously from a satellite with low distortion. The mapping of extraterrestrial bodies has resulted in the use of standard projections in completely new settings. Several other projections which have not been used by The USGS are frequently of interest to the cartographic public.

With increased computerization, it is important to realize that rectangular coordinates for all these projections may be mathematically calculated with formulas which would have seemed too complicated in the past, but which now may be programmed routinely, especially if aided by numerical examples. A discussion of appearance, usage, and history is given together with both forward and inverse equations for each projection involved.

INTRODUCTION

The subject of map projections, either generally or specifically, has been discussed in thousands of papers and books dating at least from the time of the Greek astronomer Claudius Ptolemy (about A.D. 150), and projections are known to have been in use some three centuries earlier. Most of the widely used projections date from the 16th to 19th centuries, but scores of variations have been developed during the 20th century. In recent years, there have been several new publications of widely varying depth and quality devoted exclusively to the subject. In 1979, the USGS published *Maps, for America*, a book-length description of its maps (Thompson, 1979). The USGS has also published bulletins describing from one to three projections (Birdseye, 1929; Newton, 1985).

In spite of all this literature, there was no definitive single publication on map projections used by the USGS, the agency responsible for administering the National Mapping Program, until the first edition of Bulletin 1532 (Snyder, 1982a). The USGS had relied on map projection treatises published by the former Coast and Geodetic Survey (now the National Ocean Service). These publications did not include sufficient detail for all the major projections now used by the USGS and others. A widely used and outstanding treatise of the Coast and Geodetic Survey (Deetz and Adams, 1934), last revised in 1945, only touches upon the Transverse Mercator, now a commonly used projection for preparing maps. Other projections such as the Bipolar Oblique Conic Conformal, the Miller Cylindrical, and the Van der Grinten, were just being developed, or, if older, were seldom used in 1945. Deetz and Adams predated the extensive use of the computer and pocket calculator, and, instead, offered extensive tables for plotting projections with specific parameters.

Another classic treatise from the Coast and Geodetic Survey was written by Thomas (1952) and is exclusively devoted to the five major conformal projections. It emphasizes derivations with a summary of formulas and of the history of these projections, and is directed toward the skilled technical user. Omitted are tables, graticules, or numerical examples.

In USGS Bulletin 1532 the author undertook to describe each projection which has been used by the USGS sufficiently to permit the skilled, mathematically oriented cartographer to use the projection in detail. The descriptions were also arranged so as to enable a lay person interested in the subject to learn as much as desired about the principles of these projections without being overwhelmed by mathematical detail. Deetz and Adams' (1934) work set an excellent example in this combined approach.

While Bulletin 1532 was deliberately limited to map projections used by the USGS, the interest in the bulletin has led to expansion in the form of this professional paper, which includes several other map projections frequently seen in atlases and geography texts. Many tables of rectangular or polar coordinates have been included for conceptual purposes. For values between points, formulas should be used, rather than interpolation. Other tables list definitive parameters for use in formulas. A glossary as such is omitted, since such definitions tend to be oversimplified by nature. The reader is referred to the index instead to find a more complete description of a given term.

The USGS, soon after its official inception in 1879, apparently chose the Polyconic projection for its mapping program. This projection is simple to construct and had been promoted by the Survey of the Coast, as it was then called, since Ferdinand Rudolph Hassler's leadership of the early 1800's. The first published USGS topographic "quadrangles," or maps bounded by two meridians and two parallels, did not carry a projection name, but identification as "Polyconic projection" was added to later editions. Tables of coordinates published by the USGS appeared in 1904, and the Polyconic was the only projection mentioned by Beaman (1928, p. 167).

Mappers in the Coast and Geodetic Survey, influenced in turn by military and civilian mappers of Europe, established the State Plane Coordinate System in the 1930's. This system involved the Lambert Conformal Conic projection for States of larger east-west extension and the Transverse Mercator for States which were longer from north to south. In the late 1950's, the USGS began changing quadrangles from the Polyconic to the projection used in the State Plane Coordinate System for the principal State on the map. The USGS also adopted the Lambert for its series of State base maps.

As the variety of maps issued by the USGS increased, a broad range of projections became important: The Polar Stereographic for the map of Antarctica, the Lambert Azimuthal Equal-Area for maps of the Pacific Ocean, and the Albers Equal-Area Conic for the *National Atlas* (USGS, 1970) maps of the United States. Several other projections have been used for other maps in the *National Atlas,* for tectonic maps, and for grids in the panhandle of Alaska. The mapping of extraterrestrial bodies, such as the Moon, Mars, and Mercury, involves old projections in a completely new setting. Perhaps the first projection to be originated within the USGS is the Space Oblique Mercator for continuous mapping using imagery from artificial satellites.

It is hoped that this expanded study will assist readers to understand better not only the basis for maps issued by the USGS, but also the principles and formulas for computerization, preparation of new maps, and transference of data between maps prepared on different projections.

MAP PROJECTIONS-GENERAL CONCEPTS

1. CHARACTERISTICS OF MAP PROJECTIONS

The general purpose of map projections and the basic problems encountered have been discussed often and well in various books on cartography and map projections. (Robinson, Sale, Morrison, and Muehrcke, 1984; Steers, 1970; and Greenhood, 1964, are among later editions of earlier standard references.) Every map user and maker should have a basic understanding of projections, no matter how much computers seem to have automated the operations. The concepts will be concisely described here, although there are some interpretations and formulas that appear to be unique.

For almost 500 years, it has been conclusively established that the Earth is essentially a sphere, although a number of intellectuals nearly 2,000 years earlier were convinced of this. Even to the scholars who considered the Earth flat, the skies appeared hemispherical, however. It was established at an early date that attempts to prepare a flat map of a surface curving in all directions leads to distortion of one form or another.

A map projection is a systematic representation of all or part of the surface of a round body, especially the Earth, on a plane. This usually includes lines delineating meridians and parallels, as required by some definitions of a map projection, but it may not, depending on the purpose of the map. A projection is required in any case. Since this cannot be done without distortion, the cartographer must choose the characteristic which is to be shown accurately at the expense of others, or a compromise of several characteristics. If the map covers a continent or the Earth, distortion will be visually apparent. If the region is the size of a small town, distortion may be barely measurable using many projections, but it can still be serious with other projections. There is literally an infinite number of map projections that can be devised, and several hundred have been published, most of which are rarely used novelties. Most projections may be infinitely varied by choosing different points on the Earth as the center or as a starting point.

It cannot be said that there is one "best" projection for mapping. It is even risky to claim that one has found the "best" projection for a given application, unless the parameters chosen are artificially constricting. A carefully constructed globe is not the best map for most applications because its scale is by necessity too small. A globe is awkward to use in general, and a straightedge cannot be satisfactorily used on one for measurement of distance.

The details of projections discussed in this book are based on perfect plotting onto completely stable media. In practice, of course, this cannot be achieved. The cartographer may have made small errors, especially in hand-drawn maps, but a more serious problem results from the fact that maps are commonly plotted and printed on paper, which is dimensionally unstable. Typical map paper can expand over 1 percent with a 60 percent increase in atmospheric humidity, and the expansion coefficient varies considerably in different directions on the same sheet. This is much greater than the variation between common projections on large scale quadrangles, for example. The use of stable plastic bases for maps is recommended for precision work, but this is not always feasible, and source maps may be available only on paper, frequently folded as well. On large-scale maps, such as topographic quadrangles, measurement on paper maps is facilitated with rectangular grid overprints, which expand with the paper. Grids are discussed later in this book.

The characteristics normally considered in choosing a map projection are as follows:

1. *Area*.-Many map projections are designed to be *equal-area,* so that a coin of any size, for example, on one part of the map covers exactly the same area of the actual Earth as the same coin on any other part of the map. Shapes, angles, and scale must be distorted on most parts of such a map, but there are usually some parts of an equal-area map which are designed to retain these characteristics correctly, or very nearly so. Less common terms used for equal-area projections are *equivalent*, *homolographic*, or *homalographic* (from the Greek *homalos* or *homos* ("same") and *graphos ("write")); authalic* (from the Greek *autos* ("same") and *ailos* ("area")), and *equiareal*.

2. *Shape*.-Many of the most common and most important projections are *conformal* or *orthomorphic* (from the Greek *orthos* or "straight" and *morphe* or "shape"), in that normally the relative local angles about every point on the map are shown correctly. (On a conformal map of the entire Earth there are usually one or more "singular" points at which local angles are still distorted.) Although a large area must still be shown distorted in shape, its small features are shaped essentially correctly. Conformality applies on a point or infinitesimal basis, whereas an equal-area map projection shows areas correctly on a finite, in fact mapwide basis. An important result of conformality is that the local scale in every direction around any one point is constant. Because local angles are correct, meridians intersect parallels at right (90^o) angles on a conformal projection, just as they do on the Earth. Areas are generally enlarged or reduced throughout the map, but they are correct along certain lines, depending on the projection. Nearly all large-scale maps of the Geological Survey and other mapping agencies throughout the world are now prepared on a conformal projection. No map can be both equal-area and conformal.

While some have used the term *aphylactic* for all projections which are neither equal-area nor conformal (Lee, 1944), other terms have commonly been used to describe special characteristics:

3. *Scale*.-No map projection shows scale correctly throughout the map, but there are usually one or more lines on the map along which the scale remains true. By choosing the locations of these lines properly, the scale errors elsewhere may be minimized, although some errors may still be large, depending on the size of the area being mapped and the projection. Some projections show true scale between one or two points and every other point on the map, or along every meridian. They are called *equidistant* projections.

4. *Direction*.-While conformal maps give the relative local directions correctly at any given point, there is one frequently used group of map projections, called *azimuthal* (or *zenithal),* on which the directions or azimuths of all points on the map are shown correctly with respect to the center. One of these projections is also equal-area, another is conformal, and another is equidistant. There are also projections on which directions from two points are correct, or on which directions from all points to one or two selected points are correct, but these are rarely used.

5. *Special characteristics*.-Several map projections provide special characteristics that no other projection provides. On the Mercator projection, all rhumb lines, or lines of constant direction, are shown as straight lines. On the Gnomonic projection, all great circle paths-the shortest routes between points on a sphere shown as straight lines. On the

Stereographic, all small circles, as well as great circles, are shown as circles on the map. Some newer projections are specially designed for satellite mapping. Less useful but mathematically intriguing projections have been designed to fit the sphere conformally into a square, an ellipse, a triangle, or some other geometric figure.

6. Method of construction.-In the clays before ready access to computers and plotters, ease of construction was of greater importance. With the advent of computers and even pocket calculators, very complicated formulas can be handled almost as routinely as simple projections in the past.

While the above six characteristics should ordinarily be considered in choosing a map projection, they are not so obvious in recognizing a projection. In fact, if the region shown on a map is not much larger than the United States, for example, even a trained eve cannot often distinguish whether the map is equal-area or conformal. It is necessary to make measurements to detect small differences in spacing or location of meridians and parallels, or to make other tests. The type of construction of the map projection is more easily recognized with experience, if the projection falls into one of the common categories.

There are three types of developable[1] surfaces onto which most of the map projections used by the USGS are at least partially geometrically projected. They are the cylinder, the cone, and the plane. Actually all three are variations of the cone. A cylinder is a limiting form of a cone with an increasingly sharp point or apex. As the cone becomes flatter, its limit is a plane.

If a cylinder is wrapped around the globe representing the Earth (see fig. 1), so that its surface touches the Equator throughout its circumference, the meridians of longitude may be projected onto the cylinder as equidistant straight lines perpendicular to the Equator, and the parallels of latitude marked as lines parallel to the Equator, around the circumference of the cylinder and mathematically spaced for certain characteristics. For some cases, the parallels may also be projected geometrically from a common point onto the cylinder, but in the most common cases they are not perspective. When the cylinder is cut along some meridian and unrolled, a cylindrical projection with straight meridians and straight parallels results. The Mercator projection is the best-known example, and its parallels must be mathematically spaced.

If a cone is placed over the globe, with its peak or apex along the polar axis of the Earth and with the surface of the cone touching the globe along some particular parallel of latitude, a conic (or conical) projection can be produced. This time the meridians are projected onto the cone as equidistant straight lines radiating from the apex, and the parallels are marked as lines around the circumference of the cone in planes perpendicular to the Earth's axis, spaced for the desired characteristics. The parallels may not be projected geometrically for any useful conic projections. When the cone is cut along a meridian, unrolled, and laid flat, the meridians remain straight radiating lines, but the parallels are now circular arcs centered on the apex. The angles between meridians are shown smaller than the true angles.

A plane tangent to one of the Earth's poles is the basis for polar azimuthal projections. In this case, the group of projections is named for the function, not the plane, since all

[1] A developable surface is one that can be transformed to a plane without distortion.

common tangent-plane projections of the sphere are azimuthal. The meridians are projected as straight lines radiating from a point, but they are spaced at their true angles instead of the smaller angles of the conic projections. The parallels of latitude are complete circles, centered on the pole. On some important azimuthal projections, such as the Stereographic (for the sphere), the parallels are geometrically projected from a common point of perspective; on others, such as the Azimuthal Equidistant, they are nonperspective.

The concepts outlined above may be modified in two ways, which still provide cylindrical, conic, or azimuthal projections (although the azimuthal retain this property precisely only for the sphere).

The cylinder or cone may be secant to or cut the globe at two parallels instead of being tangent to just one. This conceptually provides two standard parallels; but for most conic projections this construction is not geometrically correct. The plane may likewise cut through the globe at any parallel instead of touching a pole, but this is only useful for the Stereographic and some other perspective projections. The axis of the cylinder or cone can have a direction different from that of the Earth's axis, while the plane may be tangent to a point other than a pole (fig.1). This type of modification leads to important oblique, transverse, and equatorial projections, in which most meridians and parallels are no longer straight lines or arcs of circles. What were standard parallels in the normal orientation now become standard lines not following parallels of latitude.

Other projections resemble one or another of these categories only in some respects. There are numerous interesting pseudocylindrical (or "false cylindrical") projections. They are so called because latitude lines are straight and parallel, and meridians are equally spaced, as on cylindrical projections, but all meridians except the central meridian are curved instead of straight. The Sinusoidal is a frequently used example. Pseudoconic projections have concentric circular arcs for parallels, like conics, but meridians are curved; the Bonne is the only common example. Pseudoazimuthal projections are very rare; the polar aspect has concentric circular arcs for parallels, and curved meridians. The Polyconic projection is projected onto cones tangent to each parallel of latitude, so the meridians are curved, not straight. Still others are more remotely related to cylindrical, conic, or azimuthal projections, if at all.

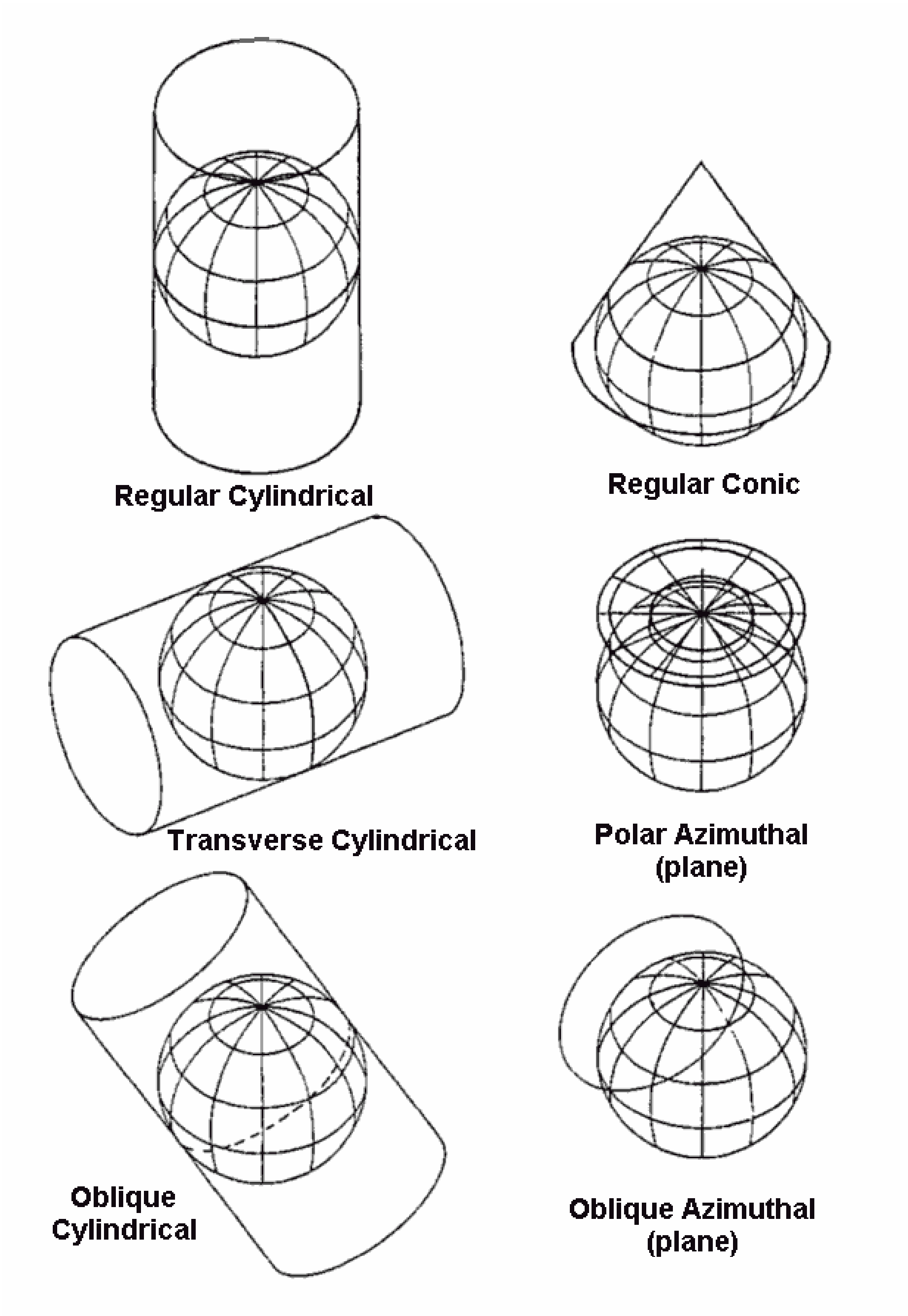


FIGURE 1.-Projection of the Earth onto the three major surfaces. In a few cases, projection is geometric, but in most cases the projection is mathematical to achieve certain features.

2. LONGITUDE AND LATITUDE

To identify the location of points on the Earth, a graticule or network of longitude and latitude lines has been superimposed on the surface. They are commonly referred to as meridians and parallels, respectively. The concept of latitudes and longitudes was originated early in recorded history by Greek and Egyptian scientists, especially the Greek astronomer Hipparchus (2nd century, B. C.). Claudius Ptolemy further formalized the concept (Brown, 1949, p. 50, 52, 68).

PARALLELS OF LATITUDE

Given the North and South Poles, which are approximately the ends of the axis about which the Earth rotates, and the Equator, an imaginary line halfway between the two poles, the parallels of latitude are formed by circles surrounding the Earth and in planes parallel with that of the Equator. If circles are drawn equally spaced along the surface of the sphere, with 90 spaces from the Equator to each pole, each space is called a degree of latitude. The circles are numbered from 0° at the Equator to 90° North and South at the respective poles. Each degree is subdivided into 60 minutes and each minute into 60 seconds of arc.

For 2,000 years, measurement of latitude on the Earth involved one of two basic astronomical methods. The instruments and accuracy, but not the principle, were gradually improved. By day, the angular height of the Sun above the horizon was measured. By night, the angular height of stars, and especially the current pole star, was used. With appropriate angular conversions and adjustments for time of day and season, the latitude was obtained.. The measuring instruments included devices known as the cross-staff, astrolabe, back-staff, quadrant, sextant, and octant, ultimately equipped with telescopes. They were supplemented with astronomical tables called almanacs, of increasing complication and accuracy. Finally, beginning in the 18th century, the use of triangulation in geodetic surveying meant that latitude on land could be determined with high precision by using the distance from other points of known latitude. Thus measurement of latitude, unlike that of longitude, was an evolutionary development almost throughout recorded history (Brown, 1949, p. 180-207).

MERIDIANS OF LONGITUDE

Meridians of longitude are formed with a series of imaginary lines, all intersecting at both the North and South Poles, and crossing each parallel of latitude at right angles, but striking the Equator at various points. If the Equator is equally divided into 360 parts, and a meridian passes through each mark, 360 degrees of longitude result. These degrees are also divided into minutes and seconds. While the length of a degree of latitude is always the same on a sphere, the lengths of degrees of longitude vary with the latitude (see fig. 2). At the Equator on the sphere, they are the same length as the degree of latitude, but elsewhere they are shorter.

There is only one location for the Equator and poles which serve as references for counting degrees of latitude, but there is no natural origin from which to count degrees of longitude, since all meridians are identical in shape and size. It thus becomes necessary to choose arbitrarily one meridian as the starting point, or prime meridian. There have been many prime meridians in the course of history, swayed by national pride and international influence. For over 150 years, France officially used the meridian through Ferro, an island of the Canaries. Eighteenth century maps of the American colonies often show longitude from London or Philadelphia. During the 19th century, boundaries of new States were described with longitudes west of a meridian through Washington, D.C., 77°03' 02.3" west of the Greenwich (England) Prime Meridian

(Van Zandt, 1976, p. 3). The latter was increasingly referenced, especially on seacharts due to the proliferation of those of British origin. In 1884, the International Meridian Conference, meeting in Washington, agreed to adopt the "meridian passing through the center of the transit instrument at the Observatory of Greenwich as the initial meridian for longitude," resolving that "from this meridian longitude shall be counted in two directions up to 180 degrees, east longitude being plus and west longitude minus" (Brown, 1949, p. 283, 297).

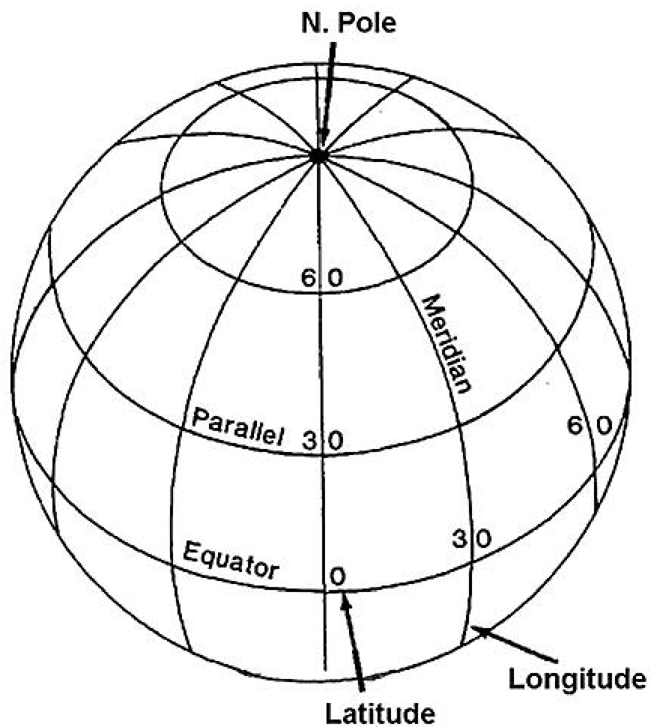


FIGURE 2.-Meridians and parallels on the sphere.

The choice of the prime meridian is arbitrary and may be stated in simple terms. The accurate measurement of the difference in longitude at sea between two points, however, was unattainable for centuries, even with a precision sufficient for the times. When extensive transatlantic exploration from Europe began with the voyages of Christopher Columbus in 1492, the inability to measure east-west distance led to numerous shipwrecks with substantial loss of lives and wealth. Seafaring nations beginning with Spain offered sizable rewards for the invention of satisfactory methods for measuring longitude. It finally became evident that a portable, dependable clock was needed, so that the height of the Sun or stars could be related to the time in order to determine longitude. The study of the pendulum by Galileo, the invention of the pendulum clock by Christian Huygens in 1656, and Robert Hooke's studies of the use of springs in watches in the 1660's provided the basic instrument, but it was not until John Harrison of England responded to his country's substantial reward posted in 1714 that the problem was solved. For five decades, Harrison devised successively more reliable versions of a marine chronometer, which were tested at sea and gradually accepted by the Board of Longitude in painstaking steps from 1765 to 1773. Final compensation required intervention by the King and Parliament (Brown, 1949, p. 208-240; Quill, 1966).

Thus a major obstacle to accurate mapping was overcome. On land, the measurement of longitude lagged behind that of latitude until the development of the clock and the spread of geodetic triangulation in the 18th century made accuracy a reality. Electronic means of measuring distance and angles in the mid- to late-20th century have redefined the meaning of accuracy by orders of magnitude.

CONVENTIONS IN PLOTTING

When constructing meridians on a map projection, the central meridian, usually a straight line, is frequently taken to be a starting point or 0° longitude for calculation purposes. When the map is completed with labels, the meridians are marked with respect to the Greenwich Prime Meridian. The formulas in this book are arranged so that Greenwich longitude may be used directly. All formulas herein use the convention of positive east longitude and north latitude, and negative west longitude and south latitude. Some published tables and formulas elsewhere use positive west longitude, so the reader is urged to use caution in comparing values.

GRIDS

Because calculations relating latitude and longitude to positions of points on a given map can become quite involved, rectangular grids have been developed for the use of surveyors. In this way, each point may be designated merely by its distance from two perpendicular axes on the flat map. The Y axis normally coincides with a chosen central meridian, y increasing north. The X axis is perpendicular to the Y axis at a latitude of origin on the central meridian, with x increasing east. Frequently x and y coordinates are called "eastings" and "northings," respectively, and to avoid negative coordinates may have "false eastings" and "false northings" added.

The grid lines usually do not coincide with any meridians and parallels except for the central meridian and the Equator. Of most interest in the United States are two grid systems: The Universal Transverse Mercator (UTM) Grid is described on p. 57, and the State Plane Coordinate System (SPCS) is described on p. 51. Preceding the UTM was the World Polyconic Grid (WPG), used until the late 1940's and described on p. 127.

Grid systems are normally divided into zones so that distortion and variation of scale within any one zone is held below a preset level. The type of boundaries between grid zones varies. Zones of the WPG and the UTM are bounded by meridians of longitude, but for the SPCS State and county boundaries are used. Some grid boundaries in other countries are defined by lines of constant grid value using a local or an adjacent grid as the basis. This adjacent grid may in turn be based on a different projection and a different reference ellipsoid. A common boundary for non-U.S. offshore grids is an ellipsoidal rhumb line, or line of constant direction on the ellipsoid (see p. 46); the ellipsoidal geodesic, or shortest route (see p.199) is also used. The plotting of some of these boundaries can become quite complicated (Clifford J. Mugnier, pers. comm., 1985).

3. THE DATUM AND THE EARTH AS AN ELLIPSOID

For many maps, including nearly all maps in commercial atlases, it may be assumed that the Earth is a sphere. Actually, it is more nearly an oblate ellipsoid of revolution, also called an oblate spheroid. This is an ellipse rotated about its shorter axis. The flattening of the ellipse for the Earth is only about one part in three hundred; but it is sufficient to become a necessary part of calculations in plotting accurate maps at a scale of 1:100,000 or larger, and is significant even for 1:5,000,000-scale maps of the United States, affecting plotted shapes by up to 2/3 percent (see p. 27). On small-scale maps, including single-sheet world maps, the oblateness is negligible. Formulas for both the sphere and ellipsoid will be discussed in this book wherever the projection is used or is suitable in both forms.

The Earth is not an exact ellipsoid, and deviations from this shape are continually evaluated. The *geoid is* the name given to the shape that the Earth would assume if it were all measured at mean sea level. This is an undulating surface that varies not more than about a hundred meters above or below a well-fitting ellipsoid, a variation far less than the ellipsoid varies from the sphere. It is important to remember that elevations and contour lines on the Earth are reported relative to the geoid, not the ellipsoid. Latitude, longitude, and all plane coordinate systems, on the other hand, are determined with respect to the ellipsoid.

The choice of the reference ellipsoid used for various regions of the Earth has been influenced by the local geoid, but large-scale map projections are designed to fit the reference ellipsoid, not the geoid. The selection of constants defining the shape of the reference ellipsoid has been a major concern of geodesists since the early 18th century. Two geometric constants are sufficient to define the ellipsoid itself. They are normally expressed either as (1) the semimajor and semiminor axes (or equatorial and polar radii, respectively), (2) the semimajor axis and the flattening, or (3) the semimajor axis and the eccentricity. These pairs are directly interchangeable. In addition, recent satellite-measured reference ellipsoids are defined by the semimajor axis, geocentric gravitational constant, and dynamical form factor, which may be converted to flattening with formulas from physics (Lauf, 1983, p. 6).

In the early 18th century, Isaac Newton and others concluded that the Earth should be slightly flattened at the poles, but the French believed the Earth to be egg-shaped as the result of meridian measurements within France. To settle the matter, the French Academy of Sciences, beginning in 1735, sent expeditions to Peru and Lapland to measure meridians at widely separated latitudes. This established the validity of Newton's conclusions and led to numerous meridian measurements in various locations, especially during the 19th and 20th centuries; between 1799 and 1951 there were 26 determinations of dimensions of the Earth.

The identity of the ellipsoid used by the United States before 1844 is uncertain, although there is reference to a flattening of 1/302. The Bessel ellipsoid of 1841 (see table 1) was used by the Coast Survey from 1844 until 1880, when the bureau adopted the 1866 evaluation by the British geodesist Alexander Ross Clarke using measurements of meridian arcs in western Europe, Russia, India, South Africa, and Peru (Shalowitz, 1964, p. 117-118; Clarke and Helmert, 1911, p. 807-808). This resulted in an adopted equatorial radius of 6,378,206.4 m and a polar radius of 6,356,583.8 m, or an approximate flattening of 1/294.9787.

The Clarke 1866 ellipsoid (the year should be included since Clarke is also known for ellipsoids of 1858 and 1880) has been used for all of North America until a change which is currently underway, as described below.

In 1909 John Fillmore Hayford reported calculations for a reference ellipsoid from U.S. Coast and Geodetic Survey measurements made entirely within the United States. This was adopted by the International Union of Geodesy and Geophysics (IUGG) in 1924, with a flattening of exactly 1/297 and a semimajor axis of exactly 6,378,388 m. This is therefore called the International or the

TABLE *1.-Some official ellipsoids in use throughout the world*[1]

	Date	Equatorial Radius, *a* meters	Polar Radius *b*, meters	Flattening *f*	Use
GRS 80[2] --------	1980	6,378,137*	6,356,752.3	1/298.257	Newly adopted
WGS 72[3]-------	1972	6,378,135*	6,356,750.5	1/298.26	NASA; Dept. of Defense; oil companies
Australian -----	1965	6,378,160*	6,356,774.7	1/298.25*	Australia
Krasovsky -----	1940	6,378,245*	6,356,863.0	1/298.3*	Soviet Union
Internat'l ------- Hayford -------	1924 1909	6,378,388*	6,356,911.9	1/297*	Remainder of the World[+]
Clarke4 ---------	1880	6,378,249.1	6,356,514.9	1/293.46**	Most of Africa; France
Clarke----------	1866	6,378,206.4*	6,356,583.8*	1/294.98	North America; Philip pines
Airy 4 --------------	1830	6,377,563.4	6,356,256.9	1/299.32**	Great Britain
Bessel ----------	1841	6,377,397.2	6,356,079.0	1/299.15**	Central Europe; Chile; Indonesia
Everest' ------	1830	6,377,276.3	6,356,075.4	1/300.80**	India; Burma; Pakistan; Afghan.; Thai land; etc.

Values are shown to accuracy in excess significant figures, to reduce computational confusion.
[1] Maling, 1973, p. 7; Thomas, 1970, p. 84; Army, 1973, p. 4, end map; Colvocoresses, 1969. p. 33; World Geodetic, 1974.
[2] Geodetic Reference System. Ellipsoid derived from adopted model of Earth. WGS 84 has same dimensions within accuracy shown.
[3] World Geodetic System. Ellipsoid derived from adopted model of Earth.
[4] Also used in some regions with various modified constants.
* Taken as exact values. The third number (where two are asterisked) is derived using the following relationships: $b = a(1-f)$; $f = 1 - b/a$. *Where* only one is asterisked (for 1972 and 1980), certain physical constants not shown are taken as exact, but *f* as shown is the adopted value.
** Derived from *a* and *b*, which are rounded off as shown after conversions from lengths in feet.
[+]Other than regions listed elsewhere in column, or some smaller areas.

Hayford ellipsoid, and is used in many parts of the world, but it was not adopted for use in North America, in part because of all the work already accomplished using the older datum and ellipsoid (Brown, 1949, p. 293; Hayford, 1909).

There are over a dozen other principal ellipsoids, however, which are still used by one or more countries (table 1). The different dimensions do not only result from varying accuracy in the geodetic measurements (the measurements of locations on the Earth), but the curvature of the Earth's surface (geoid) is not uniform due to irregularities in the gravity field.

Until recently, ellipsoids were only fitted to the Earth's shape over a particular country or continent. The polar axis of the reference ellipsoid for such a region, therefore, normally does not coincide with the axis of the actual Earth, although it is assumed to be parallel. The same applies to the two equatorial planes. The discrepancy between centers is usually a few hundred meters at most. Only satellite-determined coordinate systems, such as the WGS 72 and GRS 80 mentioned below, are considered geocentric. Ellipsoids for the latter systems represent the entire Earth more accurately than ellipsoids determined from ground measurements, but they do not generally give the "best fit" for a particular region.

The reference ellipsoids used prior to those determined by satellite are related to an "initial point" of reference on the surface to produce a datum, the name given to a smooth mathematical surface that closely fits the mean sea-level surface throughout the area of interest. The "initial point" is assigned a latitude, longitude, elevation above the ellipsoid, and azimuth to some point. Once a datum is adopted, it provides the surface to which ground control measurements are referred. The latitude and longitude of all the control points in a given area are then computed relative to the adopted ellipsoid and the adopted "initial point." The projection equations of large-scale maps must use the same ellipsoid parameters as those used to define the local datum; otherwise, the projections will be inconsistent with the ground control.

The first official geodetic datum in the United States was the New England Datum, adopted in 1879. It was based on surveys in the eastern and northeastern states and referenced to the Clarke Spheroid of 1866, with triangulation station Principio, in Maryland, as the origin. The first transcontinental arc of triangulation was completed in 1899, connecting independent surveys along the Pacific Coast. In the intervening years, other surveys were extended to the Gulf of Mexico. The New England Datum was thus extended to the south and west without major readjustment of the surveys in the east. In 1901, this expanded network was officially designated the United States Standard Datum, and triangulation station Meades Ranch. in Kansas, was the origin. In 1913, after the geodetic organizations of Canada and Mexico formally agreed to base their triangulation networks on the United States network, the datum was renamed the North American Datum.

By the mid-1920's, the problems of adjusting new surveys to fit into the existing network were acute. Therefore, during the 5-year period 1927-1932 all available primary data were adjusted into a system now known as the North American 1927 Datum."" The coordinates of station Meades Ranch were not changed but the revised coordinates of the network comprised the North American 1927 Datum (National Academy of Sciences, 1971, p. 7).

Satellite data have provided geodesists with new measurements to define the best Earth-fitting ellipsoid and for relating existing coordinate systems to the Earth's center of mass. U.S. military efforts produced the World Geodetic System 1966 and 1972 (WGS 66 and WGS 72). The National Geodetic Survey is planning to replace the North American 1927 Datum with a new datum, the North American Datum 1983 (NAD 83), which is Earth-centered based on both satellite and terrestrial data. The IUGG in 1980 adopted a new model of the Earth called the Geodetic Reference System (GRS) 80, from which is derived an ellipsoid which has been adopted for the new North American datum. As a result, the latitude and longitude of almost every point in North America will change slightly, as well as the rectangular coordinates of a given latitude and longitude on a map projection. The difference can reach 300 m. U.S. military agencies are developing a worldwide datum called WGS 84, also based on GRS 80, but with slight differences. For Earth-centered datums, there is no single "origin" like Meades Ranch on the surface. The center of the Earth is in a sense the origin.

For the mapping of other planets and natural satellites, only Mars is treated as an ellipsoid. Other bodies are taken as spheres (table 2), although some irregular satellites have been treated as triaxial ellipsoids and are "mapped" orthographically.

In most map projection formulas, some form of the eccentricity e is used, rather than the flattening *f*. The relationship is as follows:

$$e^2 = 2f - f^2, \quad or \quad f = 1 - (1 - e^2)^{1/2}$$

For the Clarke 1866, e^2 is 0.006768658. For the GRS 80, e^2 is 0.0066943800.

TABLE *2.-Official figures for extraterrestrial mapping*

[(From Davies. et al., 1983: Davies. Private commun., 1985.) Radius of Moon chosen so that all elevations are positive. Radius of Mars is based on a level of 6.1 millibar atmospheric pressure: Mars has both positive and negative elevations.]

Body	Equatorial radius *a** (kilometers)
Earth's Moon	1,738.0
Mercury	2,439.0
Venus	6,051.0
Mars	3,393.4*
Galilean satellites of Jupiter	
Io	1,815
Europa	1,569
Ganymede	2,631
Callisto	2,400
Satellites of Saturn	
Mimas	198
Enceladus	253
Tethys	525
Dione	560
Rhea	765
Titan	2,575
Iapetus	725
Satellites of Uranus	
Ariel	665
Umbriel	555
Titania	800
Oberon	815
Miranda	250
Satellite of Neptune	
Triton	1,600

* Above bodies are taken as spheres except for Mars, an ellipsoid with eccentricity *e* of 0.101929. Flattening, $1 - (1 - e^2)^{1/2}$. Unlisted satellites are taken as triaxial ellipsoids, or mapping is not expected in the near future. Mimas and Enceladus have also been given ellipsoidal parameters, but not for mapping.

[Omitted Text]

4. SCALE VARIATION AND ANGULAR DISTORTION

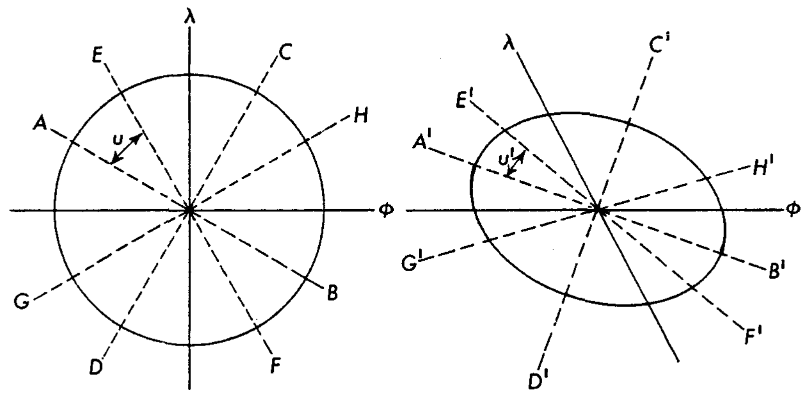
Since no map projection maintains correct scale throughout, it is important to determine the extent to which it varies on a map. On a world map, qualitative distortion is evident to an eye familiar with maps, after noting the extent to which landmasses are improperly sized or out of shape, and the extent to which meridians and parallels do not intersect at right angles, or are not spaced uniformly along a given meridian or given parallel. On maps of countries or even of continents, distortion may not be evident to the eye, but it becomes apparent upon careful measurement and analysis.

TISSOT'S INDICATRIX

In 1859 and 1881, Nicolas Auguste Tissot published a classic analysis of the distortion which occurs on a map projection (Tissot, 1881; Adams, 1919, p. 153-163; Maling, 1973, p. 64-67). The intersection of any two lines on the Earth is represented on the flat map with an intersection at the same or a different angle. At almost every point on the Earth, there is a right angle intersection of two lines in some direction (not necessarily a meridian and a parallel) which are also shown at right angles on the map. All the other intersections at that point on the Earth will not intersect at the same angle on the map, unless the map is conformal, at least at that point. The greatest deviation from the correct angle is called w, the maximum angular deformation. For a conformal map, ω is zero. (In some texts, 2ω is used rather than ω.)

Tissot showed this relationship graphically with a special ellipse of distortion called an indicatrix. An infinitely small circle on the Earth projects as an infinitely small, but perfect, ellipse on any map projection. If the projection is conformal, the ellipse is a circle, an ellipse of zero eccentricity. Otherwise, the ellipse has a major axis and minor axis which are directly related to the scale distortion and to the maximum angular deformation.

In figure 3, the left-hand drawing shows a circle representing the infinitely small circular element, crossed by a meridian λ and parallel ϕ on the Earth. The right-hand drawing shows this same element as it may appear on a typical map projection. For general purposes, the map is assumed to be neither conformal nor equal-area. The meridian and parallel may no longer intersect at right angles, but there is a pair of axes which intersect at right angles on both Earth (*AB* and *CD*) and map (*A'B'* and *C'D'*). There is also a pair of axes that on the map (*E'F'* and *G'H'*) intersect with the greatest angular deformation compared to the corresponding axes on the Earth (*EF* and *GH*, not a right angle). The latter case has the maximum angular deformation ω. The orientation of these axes is such that $\mu + \mu' = 90^\circ$, or, for small distortions, the lines fall about halfway between *A'B'* and C'D'.



(A) *(B)*

FIGURE 3.-Tissot's Indicatrix. An infinitely small circle on the Earth (*A*) appears as an ellipse on a typical map (*B*). On a conformal map, (*B*) is a circle of the same or of a different size.

The orientation is of much less interest than the size of the deformation. If a and b, the major and minor semi-axes of the indicatrix, are known, then

$$\sin(\omega/2) = |a-b| / (a-b) \qquad (4\text{-}1)$$

If lines λ and ϕ coincide with *a* and *b*, in either order, as in cylindrical and conic projections, the calculation is relatively simple, using equations (4-2) through (4-6) given below.

Scale distortion is most often calculated as the ratio of the scale along the meridian or along the parallel at a given point to the scale at a standard point or along a standard line, which is made true to scale. These ratios are called "scale factors." That along the meridian is called *h* and along the parallel, *k*. The term "scale error" is frequently applied to (*h*-1) and (*k*-1). If the meridians and parallels intersect at right angles, coinciding with *a* and *b* in figure 3, the scale factor in any other direction at such a point will fall between *h* and *k*. Angle ω may be calculated from equation (4-1), substituting *h* and *k* in place of a and *b*. In general, however, the computation of ω is much more complicated, but is important for knowing the extent of the angular distortion throughout the map.

The formulas are given here to calculate *h, k,* and ω; but the formulas for *h* and k are applied specifically to all projections for which they are deemed useful as the projection formulas are given later. Formulas for ω for specific projections have generally been omitted.

Another term occasionally used in practical map projection analysis is "convergence" or "grid declination." This is the angle between true north and grid north (or direction of the Y axis). For regular cylindrical projections this is zero, for regular conic and polar azimuthal projections it is a simple function of longitude, and for other projections it may be determined from the projection formulas by calculus from the slope of the meridian (dy/dx) at a given latitude. It is used primarily by surveyors for fieldwork with topographic maps. Convergence is not discussed further in this work.

DISTORTION FOR PROJECTIONS OF THE SPHERE

The formulas for distortion are simplest when applied to regular cylindrical, conic (or conical), and polar azimuthal projections of the sphere. On each of these types of projections, scale is solely a function of the latitude.

Given the formulas for rectangular coordinates x and y of any cylindrical projection as functions solely of longitude X and latitude 4), respectively,

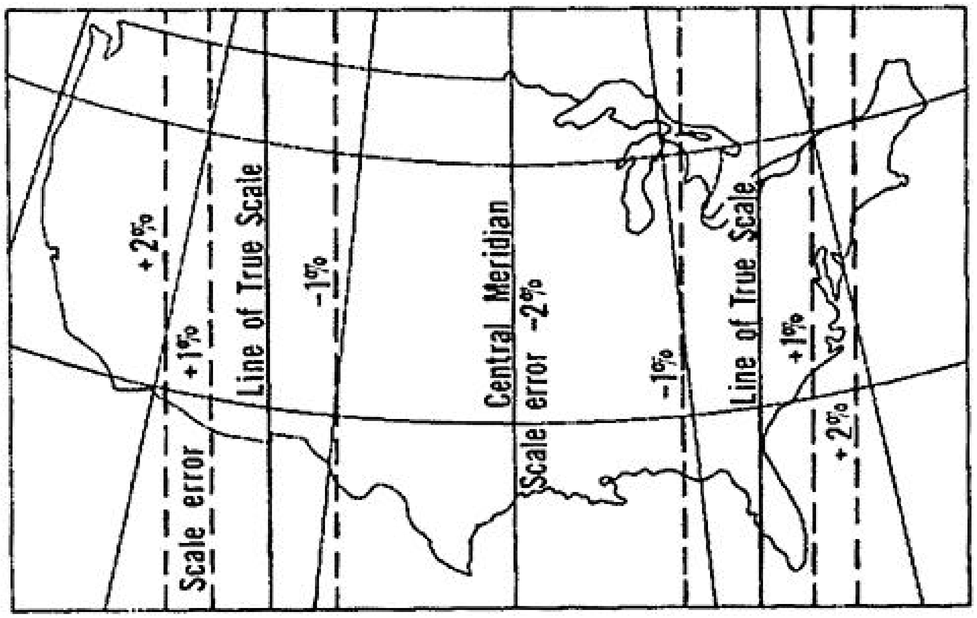
$$h = \frac{1}{R}\frac{dy}{d\phi}$$

$$k = \frac{1}{R\cos\phi}\frac{dx}{d\lambda} \qquad \text{(4-2), (4-3)}$$

Given the formulas for polar coordinates ρ and θ of any conic projection as functions solely of ϕ and λ, respectively, where *n* is the cone constant or ratio of θ to $(\lambda - \lambda_0)$,

$$h = \frac{1}{R}\frac{d\rho}{d\phi} \qquad \text{(4-4)}$$

$$k = \frac{n\rho}{R\cos\phi} \qquad \text{(4-5)}$$



Transverse Mercator Projection

FIGURE 4-Distortion patterns on common conformal map projections. The Transverse Mercator and the Stereographic are shown with reduction in scale along the central meridian or at the center of projection, respectively. If there is no reduction, there is a single line of true scale along the central meridian on the Transverse Mercator and only a point of true scale at the center of the Stereographic. The illustrations are conceptual rather than precise, since each base map projection is an identical conic.

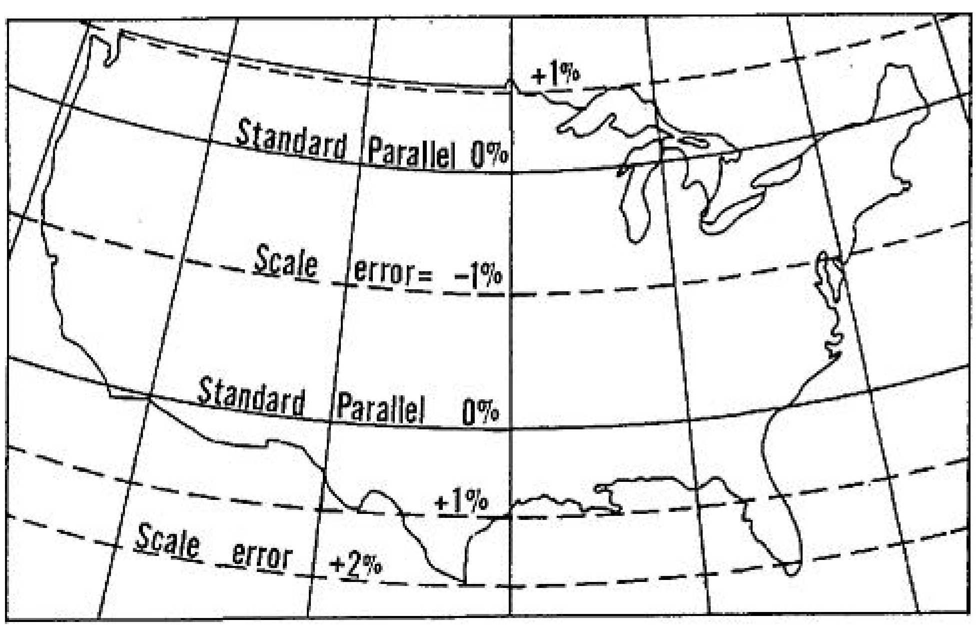


Figure 4 - Continued

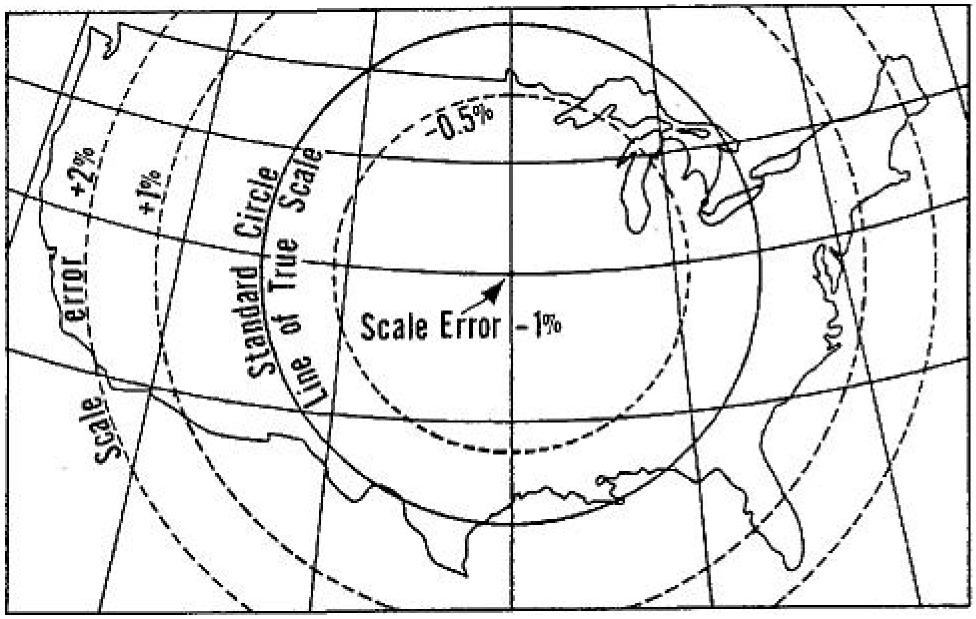


Figure 4 - Continued

[Omitted Text]

Latitude	Clarke 1866 ellipsoid		International (Hayford) ellipsoid	
(0)	1' lat.	1' long.	1° lat.	1' long.
90° ---------------------	111,699.4	0.0	111,700.0	0.0
85	111,690.7	9,735.0	111,691.4	9,735.0
80	111,665.0	19,394.4	111,665.8	19,394.5
75	111,622.9	28,903.3	111,624.0	28,903.5
70	111,565.9	38,188.2	111,567.4	38,188.5
65	111,495.7	47,177.5	111,497.7	47,177.9
60	111,414.5	55,802.2	111,417.1	55,802.8
55 ----------------------	111,324.8	63,996.4	111,327.9	63,997.3
50	111,229.3	71,698.1	111,233.1	71,699.2
45	111,130.9	78,849.2	111,135.4	78,850.5
40 ----------------------	111,032.7	85,396.1	111,037.8	85,397.7
35	110,937.6	91,290.3	110,943.3	91,292.2
30 ----------------------	110,848.5	96,488.2	110,854.8	96,490.4
25	110,768.0	100,951.9	110,774.9	100,954.3
20 ---------------------	110,698.7	104,648.7	110,706.0	104,651.4
15 ----------------------	110,642.5	107,551.9	110,650.2	107,554.8
10 ----------------------	110,601.1	109,640.7	110,609.1	109,643.7
5 -----------------------	110,575.7	110,899.9	110,583.9	110,903.1
0 -----------------------	110,567.2	111,320.7	110,575.5	111,323.9

[Omitted Text]

6. CLASSIFICATION AND SELECTION OF MAP PROJECTIONS

Because of the hundreds of map projections already published and infinite number which are theoretically possible, considerable attention has been given to classification of projections so that the user is not overwhelmed by the numbers and the variety. Generally, the proposed systems classify projections on the basis of property (equal-area, conformal, equidistant, azimuthal, and so forth), type of construction (cylindrical, conical, azimuthal, and so forth), or both. Lee (1944) proposed a combination:

Conical projections

- Cylindric
- Pseudocylindric
- Conic
- Pseudoconic
- Polyconic Azimuthal
  - Perspective
  - Nonperspective

Nonconical projections

- Retroazimuthal (not discussed here)
- Orthoapsidal (not discussed here)
- Miscellaneous

Each of these categories was further subdivided into conformal, authalic (equal-area), and aphylactic (neither conformal nor authalic), but some subdivisions have no examples. This classification is partially used in this book, as the section headings indicate, but the headings are influenced by the number of projections described in each category: Pseudocylindrical projections are included with the "miscellaneous" group, and "space map projections" are given a separate section.

Interest has been shown in some other forms of classification which are more suitable for extensive treatises. In 1962, Waldo R. Tobler provided a simple but all-inclusive proposal (Tobler, 1962). Tobler's classification involves eight categories, four for rectangular and four for polar coordinates. For the rectangular coordinates, category A includes all projections in which both x and y vary with both latitude ϕ and longitude λ, category *B* includes all in which y varies with both ϕ and λ while x is only a function of λ, C includes those projections in which x varies with both ϕ and λ while y varies only with ϕ, and *D* is for those in which x is only a function of λ and y only of ϕ. There are very few published projections in category *B*, but C is usually called pseudocylindrical, *D* is cylindrical, and *A* includes nearly all the rest which do not fit the polar coordinate categories.

Tobler's categories A to D for polar coordinates are respectively the same as those for rectangular, except that radius ρ is read for y and angle θ is read for x. The regular conic and azimuthal projections fall into category *D*, and the pseudo-conical (such as Bonne's) into C. Category A may have a few projections like A (rectangular) for which polar coordinates are more convenient than rectangular. There are no well-known projections in B (polar).

Hans Maurer's detailed map projection treatise of 1935 introduced a "Linnaean" classification with five families ("true-circular," "straight-symmetrical," "curved symmetrical," "less regular," and "combination grids," to quote a translation) subdivided into branches, subbranches, classes, groups, and orders (Maurer, 1935). As Maling says, Maurer's system "is only useful to the advanced student of the subject," but Maurer attempts for map projections what Linnaeus, the Swedish botanist, accomplished for plants and animals in the 18th century (Maling, 1973, p. 98). Other approaches have been taken by Goussinsky (1951) and Starostin (1981).

SUGGESTED PROJECTIONS

Following is a simplified listing of suggested projections. The recommendation can be directly applied in many cases, but other parameters such as the central meridian and parallel or the standard parallels must also be determined. These additional parameters are often chosen by estimation, but they can be chosen by more refined methods to reduce distortion (Snyder, 1985a, p. 93-109). In other cases a more complicated projection may be chosen because of special features in the extent of the *region* being mapped; the GS50 projection (50-State map) described in this book is an example. Some commonly used projections are not listed in this summary because it is felt that other projections are more suitable for the applications listed, which are not all-inclusive. Some of the projections listed here are not discussed elsewhere in this book.

Region mapped

1. World (Earth should be treated as a sphere)
   - A. Conformal (gross area distortion)
     - (1) Constant scale along Equator
       - Mercator
     - (2) Constant scale along meridian
       - Transverse Mercator
     - (3) Constant scale along oblique great circle
       - Oblique Mercator
     - (4) Entire Earth shown
       - Lagrange
       - August
       - Eisenlohr
   - B. Equal-Area
     - (1) Standard without interruption
       - Hammer
       - Mollweide
       - Eckert IV or VI
       - McBryde or McBryde-Thomas variations
       - Boggs Eumorphic
       - Sinusoidal
       - misc. pseudocylindricals
     - (2) Interrupted for land or ocean
       - any of above except Hammer Goode Homolosine
     - (3) Oblique aspect to group continents
       - Briesemeister
       - Oblique Mollweide
   - C. Equidistant
     - (1) Centered on pole
       - Polar Azimuthal Equidistant
     - (2) Centered on a city
       - Oblique Azimuthal Equidistant
   - D. Straight rhumb lines
     - Mercator
   - E. Compromise distortion
     - Miller Cylindrical
     - Robinson

2. Hemisphere (Earth should be treated as a sphere)
   - A. Conformal
     - Stereographic (any aspect)
   - B. Equal-Area
     - Lambert Azimuthal Equal-Area (any aspect)
   - C. Equidistant
     - Azimuthal Equidistant (any aspect)
   - D. Global look
     - Orthographic (any aspect)

3. Continent, ocean, or smaller region (Earth should be treated as a sphere for larger continents and oceans and as an ellipsoid for smaller regions, especially at a larger scale)

A. Predominant east-west extent

(1) Along Equator

Conformal: Mercator

Equal-Area: Cylindrical Equal-Area

(2) Away from Equator

Conformal: Lambert Conformal Conic

Equal-Area: Albers Equal-Area Conic

B. Predominant north-south extent

Conformal: Transverse Mercator

Equal-Area: Transverse Cylindrical Equal-Area

C. Predominant oblique extent (for example: North America, South America, Atlantic Ocean)

Conformal: Oblique Mercator

Equal-Area: Oblique Cylindrical Equal-Area

D. Equal extent in all directions (for example: Europe, Africa, Asia, Australia, Antarctica, Pacific Ocean, Indian Ocean, Arctic Ocean, Antarctic Ocean)

(1) Center at pole

Conformal: Polar Stereographic

Equal-Area: Polar Lambert Azimuthal Equal-Area

(2) Center along Equator

Conformal: Equatorial Stereographic

Equal-Area: Equatorial Lambert Azimuthal Equal-Area

(3) Center away from pole or Equator

Conformal: Oblique Stereographic

Equal-Area: Oblique Lambert Azimuthal Equal-Area

E. Straight rhumb lines (principally for oceans)

Mercator

F. Straight great-circle routes

Gnomonic (for less than hemisphere)

G. Correct scale along meridians

(1) Center at pole

Polar Azimuthal Equidistant

(2) Center along Equator

Plate Carree (Equidistant Cylindrical)

(3) Center away from pole or Equator

Equidistant Conic