Radius of the Earth - Radii Used in Geodesy

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I. Three Radii of Earth and Their Use

There are three radii that come into use in geodesy. These are a function of latitude in the ellipsoidal model of the earth. The physical radius, the distance from the center of the earth to the ellipsoid is the least used. The other two are the radii of curvature.

If you wish to convert a small difference of latitude or longitude into the linear distance on the surface of the earth, then the spherical earth equation would be,

$$dN = R\, d\phi,$$
$$dE = R\, \cos\phi\, d\lambda$$

where R is the radius of the sphere and the angle differences are in radians. The values dN and dE are distances in the surface of the sphere. For an ellipsoidal earth there is a different radius for each of the directions.

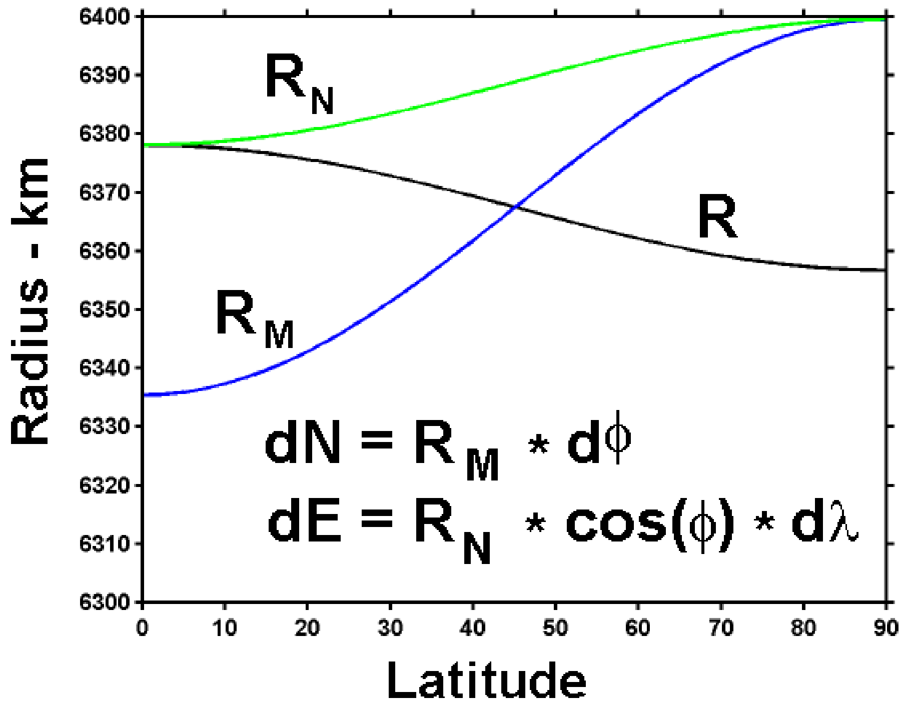
The radius used for the longitude is called the Radius of Curvature in the prime vertical. It is denoted by R_N, N, or ν, or a few other symbols. (Note that in these notes the symbol N is used for the geoid undulation.) It has a nice physical interpretation. At the latitude chosen go down the line perpendicular to the ellipsoid surface until you intersect the polar axis. This distance is R_N.

The radius used for the latitude change to North distance is called the Radius of Curvature in the meridian. It is denoted by R_M, or M, or several other symbols. It has no good physical interpretation on a figure. It is the radius of a circle that fits the earth curvature in the North-South (the meridian) at the latitude chosen.

The equations for the relation between the differential distances and angles are now:

$$dN = R_M\, d\phi,$$
$$dE = R_N\, \cos\phi\, d\lambda$$

The new radii of curvature are used in place of the simple single radius of the sphere. For the earth these are shown in the following diagram. (To be precise, these are the WGS 84 ellipsoid values.) Notice that the latitude related curvature, R_M, is largest at the poles where the earth is flattest. The radius of curvature for a straight line is infinite. The physical radius is maximum at the equator and minimum at the poles.



Earth Radius
Radii of Curvature vs Latitude

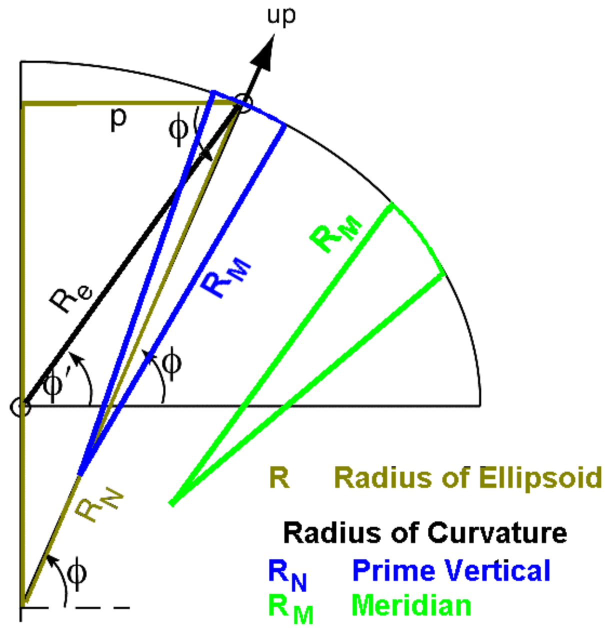
II. Diagrams of Radii of Earth

The radius of curvature for the latitude is found by extending the line perpendicular to the ellipsoid (the geodetic vertical) down until it hits the polar axis. Perpendicular is also called "normal" in mathematics. Thus the subscript "N" is used for this curvature. The subscript "M" comes from meridian, the name of the lines that run north-south on a globe or map.

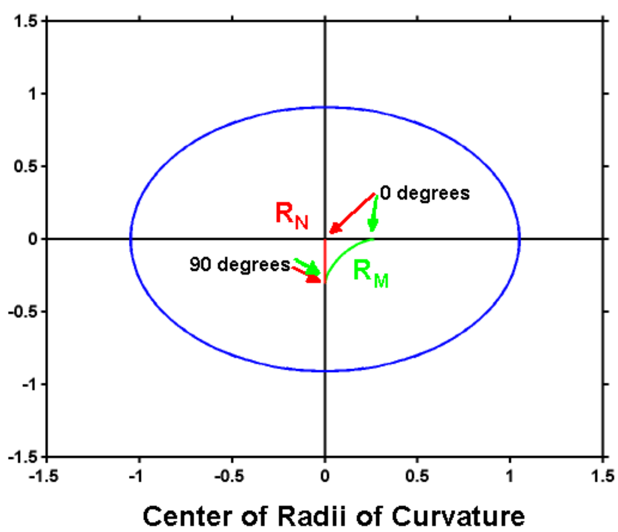
A diagram of these radii is shown below. Note that the geodetic latitude (ϕ**,** or ϕ_g**)**, is used. This should not be confused with the geocentric latitude (ϕ'**,** or ϕ_c**)**. The radius of curvature in the meridian, R_M, is shown for two different latitudes. Two lines and an arc of the circle tangent to the ellipse are shown to illustrate the origin of this radius.

The important auxiliary line, p, is included. This is the moment arm length for the rotational accelerations. It also completes a triangle that includes the geodetic latitude inside at the p - R_N intersection. Two equations for p can be derived from this latitude diagram. The equation using the geodetic latitude is more commonly used.

$$
\begin{aligned}
p &= R_N \cos\phi \\
&= R \cos\phi_c
\end{aligned}
$$



The end of the radius R_N always is on the polar axis. The end for the meridian radius, R_M, is in the quadrant on the other side of the equator. This location has no physical meaning. A diagram of the centers is shown below for reference. This was computed for a mathematical ellipse of radius about 1 and eccentricity of 0.5. For the real earth, the difference of either center from the center would not be visible on a plot.



III. Equations for Radii of Earth

The equations for these radii are given in the following table. A note of caution on the use of these equations is in order. Ellipses occur in geodesy and satellite work. Earth satellite work involves both. But the equations and symbols are different. In geodesy the geodetic latitude is use. This has no equivalent in satellite work. In satellite work four other angles are used. They originate at one focus or at the center. The equation for the radius of the orbit using any of these angles is different from any of the radii in the table.

Radius	Formula
Radius of Curvature: in Prime Vertical, terminated by minor axis	$R_N = \dfrac{a}{\sqrt{1-e^2\sin^2\phi}}$
Radius of Curvature: in Meridian	$R_M = \dfrac{a(1-e^2)}{\left(1-e^2\sin^2\phi\right)^{3/2}} = R_N\dfrac{1-e^2}{1-e^2\sin^2\phi}$
Radius of ellipsoid	$R = \dfrac{a\sqrt{[(1-e^2)^2\sin^2\phi+\cos^2\phi]}}{\sqrt{1-e^2\sin^2\phi}}$
Radius of Curvature: At azimuth α	$\dfrac{1}{R_\alpha} = \dfrac{\cos^2\alpha}{R_M} + \dfrac{\sin^2\alpha}{R_N}$

The formula for the radius of curvature at arbitrary azimuth points up that the fundamental mathematical quantity is the inverse of these radii, which are simply called curvatures.

The value of the eccentricity, e, used in these equations is given by

$$e = \sqrt{1-\frac{b^2}{a^2}}$$

with a being the semi-major axis (equatorial radius) and b the semi-minor axis (polar radius) of the ellipsoid. Notice that $1 - e^2 = b^2/a^2$. The term $1 - e^2$ is common in ellipse equations. It just is the ratio of the axes squared.

The flattening, f, is defined as

$$f = 1 - \frac{b}{a}.$$

Notice that $1 - f = b/a$. It is common in geodesy to define the size and shape of an ellipsoid by giving a and f. However the eccentricity occurs more frequently in equations. The flattening is related to the eccentricity by

$$e^2 = 2f - f^2$$

IV. Other Useful Equations Involving Radii of Earth

The area of an ellipse with semi-major axis of a and semi-minor axis of b is given by:

$$A = \pi a b \ .$$

The volume of an ellipse of revolution, revolved about the semi-minor axis, is given by,

$$V = \frac{4}{3}\pi a^2 b \ .$$

It should be noted that if you want a sphere of the same volume as an ellipse of revolution you need a radius R_S given by,

$$R_S^3 = a^2 b \ .$$

The ellipsoid used in the diagram for the centers has an equivalent spherical radius of 1 and an eccentricity of 0.5.

The average radius of curvature at any latitude is the geometric mean of R_N and R_M. That is:

$$\begin{aligned} \overline{R}_c &= \frac{1}{2\pi}\int_0^{2\pi} R_\alpha \, d\alpha \ , \\ &= \sqrt{R_N R_M} \end{aligned}$$

where α is the azimuth.